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STATISTICAL, NONLINEAR, AND SOFT MATTER PHYSICS

Hydrodynamic "Memory" of Binary Fluid Mixtures

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Abstract—A theoretical analysis is presented of hydrostatic adjustment in a two-component fluid system, such as seawater stratified with respect to temperature and salinity. Both linear approximation and nonlinear problem are investigated. It is shown that scenarios of relaxation to a hydrostatically balanced state in binary fluid mixtures may substantially differ from hydrostatic adjustment in fluids that can be stratified only with respect to temperature. In particular, inviscid two-component fluids have "memory": a horizontally nonuniform disturbance in the initial temperature or salinity distribution does not vanish even at the final stage, transforming into a persistent thermohaline "trace." Despite stability of density stratification and convective stability of the fluid system by all known criteria, an initial temperature disturbance may not decay and may even increase in amplitude. Moreover, its sign may change (depending on the relative contributions of temperature and salinity to stable background density stratification). Hydrostatic adjustment may involve development of discontinuous distributions from smooth initial temperature or concentration distributions. These properties of two-component fluids explain, in particular, the occurrence of persistent horizontally or vertically nonuniform temperature and salinity distributions in the ocean, including discontinuous ones.

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1. INTRODUCTION

It is well known that binary fluid mixtures have some fluid-dynamic and thermodynamic properties that are peculiar at first glance. The most important example is double-diffusive convection: convective instability due to difference in diffusivity between two substances can develop even when the background density stratification is stable [1]. Some new nontrivial properties of such fluids were revealed in the past years (e.g., see [2–4]). In particular, it was shown that a fluid system of this kind can have a "negative heat capacity": the addition of heat can cause a decrease rather than an increase its temperature. Stronger density stratification in a stably stratified medium may entail an increase rather than a decrease in the amplitude of its response to heating, as well as in the temperature disturbance depth. Some previously unknown mechanisms of convective instability have also been discovered.

However, the unexpected properties of two-component fluids are not restricted to those mentioned above. In this paper, we show that hydrostatic adjustment in stratified inviscid two-component fluid systems can follow scenarios that are different from those characteristic of fluids that can be stratified only with respect to temperature. In systems of the latter type, a horizontally nonuniform disturbance in the initial temperature distribution decays via decay of the wave motion induced by the disturbance. These phenomena play a very important role, in particular, in geophysical fluid dynamics (in the atmosphere, ocean, and planetary interiors), making it difficult to explain the existence of certain frequently observed types of persistent nonuniform temperature distributions in the upper ocean (e.g., see [5, 6]). The analysis presented in this paper shows that horizontal disturbances of temperature and salinity distributions in a doubly stratified ocean do not vanish even in the hydrostatically balanced final state. In other words, two-component fluids "keep memory" of initial disturbances in the form of persistent horizontally nonuniform "traces" in temperature and concentration distributions. One common feature of "traces" is the development of discontinuous distributions (jumps) from smooth initial distributions. Patterns of this kind (particularly jumps in vertical distributions) are widespread in the upper layer of the ocean [7, 8]. Explanation of their origin is a problem of considerable interest.

In terms of fluid dynamics, the "memory" effect in two-component fluid systems discussed in this paper is explained by the fact that temperature and salinity are Lagrangian invariants of adiabatic flows; i.e., they are conserved in each fluid element. The integral and Lagrangian conservation laws underlying this effect are discussed in the last section.

It is commonly believed that flow and thermodynamic disturbances must decay in a stable density-stratified fluid, because vertical displacement generates a restoring force. Therefore, the following result presented below is even less obvious: disturbances of temperature and salinity can drastically increase in amplitude even against the background of arbitrary stable density stratification (in the absence of double-diffusive effects). In this sense, doubly stratified two-component fluid systems of the type discussed here are active media.

2. STATEMENT OF THE PROBLEM

We analyze the behavior of disturbances in a quiescent unbounded inviscid two-component fluid under constant gravity. To be specific, we consider stable hydrostatically balanced states of salt water stratified with respect to temperature and salinity; i.e., the overall density stratification is supposed to be stable even though the temperature and salinity fields may be unstably stratified.

Invoking a commonly used approximation [1, 9], we assume that the fluid density ρ is a linear function of perturbations of temperature T and salinity s:

$$\rho = \rho_*(1 - \alpha T + \beta s), \tag{1}$$

where ρ_* is the value of ρ corresponding to constant mean values of temperature T_* and salinity s_*, T and s denote the respective deviations from these mean values, α is the thermal expansion coefficient, and β is the coefficient of salinity compression. Using (1), we write a closed system of equations describing adiabatic fluid motion and salinity evolution:

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p - g\mathbf{k}, \quad \operatorname{div}\mathbf{u} = 0, \tag{2}$$

$$\frac{dT}{dt} = 0, \quad \frac{ds}{dt} = 0. \tag{3}$$

Here, $\mathbf{u}$ is the velocity vector with components u, v, and w in a coordinate system xyz (with positive x direction upward); p is pressure; g is the gravitational acceleration; $\mathbf{k}$ is the unit vector along the z axis; and $d/dt = \partial/\partial t + (\mathbf{u}, \nabla)$ is the total derivative operator.

The initial conditions for system (1)–(3) are written as

$$\begin{gathered} \mathbf{u}|_{t=0} = \mathbf{0}, \\ T|_{t=0} = \gamma_T z + T_i(\mathbf{x}), \quad s|_{t=0} = \gamma_s z + s_i(\mathbf{x}), \end{gathered} \tag{4}$$

where $\mathbf{x} = (x, y, z)$, and T_i and s_i are prescribed initial disturbances. As mentioned above, the vertical gradients γ_T and γ_s are supposed to be such that the background stratification is stable, which means that $\rho(z) = \rho_*(1 - \gamma z)$ and $\gamma = \alpha\gamma_T - \beta\gamma_s > 0$.

The relative contributions of temperature and salinity to the background stratification are conveniently characterized by the dimensionless parameter

$$\eta = \frac{\beta\gamma_s}{\alpha\gamma_T}. \tag{5}$$

In the case of a fluid stratified only with respect to temperature, $\eta = 0$. It is obvious that $\gamma = \alpha\gamma_T(1 - \eta)$. In certain cases, a more suitable parameter is

$$\xi = \frac{\eta}{\eta - 1} = \frac{\beta\gamma_s}{\beta\gamma_s - \alpha\gamma_T} = \frac{d\bar{\rho}_s/dz}{d(\bar{\rho}_s + \bar{\rho}_T)/dz}, \tag{5a}$$

where $d\bar{\rho}_T/dz$ and $d\bar{\rho}_s/dz$ denote the vertical density gradients due to temperature and salinity stratification, respectively; i.e., ξ can be interpreted as the relative contribution of salinity to the background density stratification.

In an unbounded stratified fluid, a horizontally nonuniform initial disturbance of the density field, $\rho_i = -\rho^*(\alpha T_i - \beta s_i)$, gives rise to wave motion, which smooths out the density distribution and decays in the long-time limit. Hydrostatic adjustment involving wave motion is most amenable to analysis in the case of small-amplitude disturbances.

3. LINEAR APPROXIMATION

Denoting small perturbations of thermodynamic variables by primes and using the Boussinesq approximation, we obtain the following linear system instead of (1)–(3):

$$\frac{\partial\mathbf{u}}{\partial t} = -\frac{1}{\rho_*}\nabla p' + g(\alpha T' - \beta s')\mathbf{k}, \quad \operatorname{div}\mathbf{u} = 0, \tag{6}$$

$$\frac{\partial T'}{\partial t} + \gamma_T w = 0, \quad \frac{\partial s'}{\partial t} + \gamma_s w = 0 \tag{7}$$

with the initial conditions

$$\mathbf{u}|_{t=0} = 0, \quad T'|_{t=0} = T_i(\mathbf{x}), \quad s'|_{t=0} = s_i(\mathbf{x}).$$

Introducing the dimensionless buoyancy

$$\sigma = -\frac{\rho'}{\rho_*} = \alpha T' - \beta s',$$

we reduce system (6), (7) to a single equation:

$$\frac{\partial^2}{\partial t^2}\Delta_3\sigma + N^2\Delta_2\sigma = 0, \tag{8}$$

where $N = \sqrt{g\gamma}$ is the Brunt–Väisälä frequency (buoyancy frequency), and Δ_3 and Δ_2 denote the three- and two-dimensional (horizontal) Laplace operators, respectively. Equation (8) plays a fundamental role in the linear theory of internal gravity waves [10] (it is generally formulated for the vertical velocity component). A general solution to the Cauchy problem for this equation was constructed in [11]. In the solution describing wave dissipation in the medium, density disturbances decay with time: $\sigma \longrightarrow 0$ as $t \longrightarrow \infty$.

In a one-component medium (with density depending only on temperature), the temperature perturbation T' obviously decays together with σ. The behavior of T' in a two-component fluid is totally different. To find the final temperature and salinity distributions $T'_{\rm f}$ and $s'_{\rm f}$ (as $t \longrightarrow \infty$), we use a local conservation law: the linearized form of a "freezing-in" equation for two-component fluids (see Section 5). This law is obtained by eliminating w from (7):

$$\frac{\partial r}{\partial t} = 0, \quad r = \gamma_s T' - \gamma_T s'. \tag{9}$$

According to (9), the field $r(\mathbf{x})$ does not vary with time:

$$\gamma_s T'_{\rm f} - \gamma_T s'_{\rm f} = \gamma_s T_i - \gamma_T s_i. \tag{10}$$

Furthermore, $\sigma = 0$, i.e., $\alpha T'_{\rm f} = \beta s'_{\rm f}$. Combining this relation with (10), we obtain

$$T'_{\rm f} = \frac{\alpha\eta T_i - \beta s_i}{\alpha(\eta - 1)}, \quad s'_{\rm f} = \frac{\alpha\eta T_i - \beta s_i}{\beta(\eta - 1)}. \tag{11}$$

These expressions show that the disturbances T' and s' do not vanish even at the final stage of hydrostatic adjustment. Their contributions compensate each other in the density field ($\sigma_{\rm f} = 0$), and they transform into a steady pattern that can be called *thermohaline trace*. Note that a "trace" of this kind in a real two-component fluid must vanish in a characteristic dissipation time. However, the dissipation time scale is much larger than the time scale of hydrostatic adjustment for disturbances characterized by relatively large length scales.

Consider the final distributions in the case of $s_i = 0$:

$$T'_{\rm f} = \frac{\eta}{\eta - 1} T_i, \quad s'_{\rm f} = \frac{\alpha}{\beta}\frac{\eta}{\eta - 1} T_i. \tag{12}$$

When $\eta = 0$ (one-component fluid), the initial distribution leaves no trace: $T'_{\rm f} = s'_{\rm f} = 0$. When $\eta \neq 0$, three qualitatively different situations are possible under the constraint $\gamma > 0$, which correspond to different values of the parameter $\xi = \eta/(\eta - 1)$.

3.1. The background temperature and salinity fields are stably stratified: $\gamma_T > 0$, $\gamma_s < 0$, $\eta < 0$. In this case, $0 < \xi < 1$; i.e., the final temperature disturbance retains the sign of the initial one and has a smaller amplitude. The decrease in disturbance amplitude in a stably stratified medium looks quite natural, even though the incomplete decay of the disturbance is a surprising result.

3.2. The temperature stratification is unstable, and the system is stabilized by stable salinity stratification: $\beta\gamma_s < \alpha\gamma_T < 0$, $\eta > 1$. In this case, $\xi > 1$; i.e., the final disturbance amplitude is always larger than the initial amplitude. Furthermore, as $\eta \longrightarrow 1 + 0$ (neutral density stratification is approached), $T'_{\rm f} \longrightarrow \infty$ ($\xi \longrightarrow \infty$). The increase in disturbance amplitude in a stably stratified two-component fluid can be explained as follows. A positive initial temperature disturbance induces an upward motion of warmer fluid parcels when $\gamma_T < 0$. Nearly neutral density stratification does not impede this motion, and a "trace" develops. The dimensionless parameter ξ characterizes a relative increase in disturbance amplitude. Note that the increase may occur against the background of any stable density stratification. This is clear from expression (5a), where the denominator $d(\bar\rho_s + \bar\rho_T)/dz$ can have any absolute value, but is always smaller than $d\bar\rho_s/dz$ in the case under consideration.

3.3. The salinity stratification is unstable, and the system is stabilized by stable temperature stratification: $\alpha\gamma_T > \beta\gamma_s > 0$, $0 < \eta < 1$. In this (most interesting) case, $\xi < 0$ and $|\xi| \longrightarrow \infty$ as $\eta \longrightarrow 1 - 0$. Thus, the final and initial disturbances have opposite signs; for example, an initial heating leads to the formation of a cool "trace," which can be much stronger than the initial temperature disturbance. At first glance, the "negative heat capacity" of a stratified two-component fluid system [2] is an unexpected effect. However, it also has a simple explanation: in the case of $\gamma_T > 0$ and nearly neutral density stratification, the developing vertical motion brings relatively cool fluid parcels from lower to upper layers.

Note also the following feature of final distributions. Expressions (11) imply that discontinuities (jumps) in the initial distributions T_i and s_i persist through the final stage. For example, even if only the temperature field is discontinuous at the initial instant, it follows from (11) that the salinity field becomes discontinuous in the course of hydrostatic adjustment. It is shown below that formation of discontinuities in initially smooth disturbances is a universal characteristic of nonlinear dynamics.

4. NONLINEAR THEORY

Now, we analyze the final structure of disturbances in a nonlinear problem. For simplicity, we consider two-dimensional motion in the xz plane. We assume that $s_i = 0$ at the initial instant and $T_i(x, z)$ is horizontally localized: $T_i \longrightarrow 0$ as $|x| \longrightarrow \infty$. The corresponding density distribution has the form

$$\rho_0(x, z) = \bar\rho(z) + \rho_i(x, z),$$

where $\bar\rho = \rho_*(1 - \gamma z)$ and $\rho_i = -\alpha\rho_* T_i$.

In the nonlinear problem, the variables ρ, T, and s are Lagrangian invariants; i.e., their values are conserved in each fluid element. This fact can be used to find their final distributions. First, we show that $\rho_f(z) = \bar{\rho}(z)$ in the balanced state corresponding to the final stage of hydrostatic adjustment; i.e., the initial density distribution is restored. Indeed, the definition of Lagrangian invariant implies that $\rho = \rho_0(x_0, z_0)$ at any $t > 0$, where x_0 and z_0 are the initial (Lagrangian) coordinates of a fluid element. Therefore, $\rho_f(z) = \rho_0(x_0, z_0)$; i.e.,

$$\rho_f(z) = \bar{\rho}(z_0) + \rho_i(x_0, z_0). \tag{13}$$

Since $\rho_i \longrightarrow 0$ and $z_0 \longrightarrow z$ as $|x_0| \longrightarrow \infty$, expression (13) reduces to $\rho_f(z) = \bar{\rho}(z)$ in this limit.

Substituting $\rho_f(z) = \bar{\rho}(z)$ into (13), we obtain

$$\rho_*(1-\gamma z) = \rho_*(1-\gamma z_0) - \alpha\rho_* T_i(x_0, z_0),$$

which yields

$$z = z_0 + \frac{\alpha}{\gamma}T_i(x_0, z_0). \tag{14}$$

Expression (14) relates the final (Eulerian) vertical coordinate z of a fluid element to its initial (Lagrangian) coordinates. If this relation is known, then an analogous relation, $x = x(x_0, z_0)$, can be found for the horizontal coordinate by using the continuity equation written in the Lagrangian variables:

$$\frac{\partial(x, z)}{\partial(x_0, z_0)} = 1. \tag{15}$$

We seek a solution of the form $x = x(x_0, z)$. Substituting this expression into Eq. (15), we obtain $\partial x/\partial x_0 = (\partial x/\partial z_0)^{-1}$ and find x by integration. Thus, (14) and (15) make up a closed system of equations for the Lagrangian coordinates of fluid elements.

Now, we can find the final temperature and salinity distributions. By the definition of Lagrangian invariant and initial conditions (4), we have

$$T_f = \gamma_T z_0 + T_i(x_0, z_0), \quad s_f = \gamma_s z_0. \tag{16}$$

These expressions are combined with the relations $x = x(x_0, z_0)$ and $z = z(x_0, z_0)$ to obtain a parametric representation of the functions $T_f = T_f(x, z)$ and $s_f = s_f(x, z)$. Substituting the expression $z_0 = z - (\alpha/\gamma)T_i$ derived from (14) and the relation $\alpha\gamma_T/\gamma = 1/(1 - \eta)$ into (16), we obtain

$$T_f = \gamma_T z + \frac{\eta}{\eta - 1}T_i(x_0, z_0),$$
$$s_f = \gamma_s z + \frac{\alpha}{\beta}\frac{\eta}{\eta - 1}T_i(x_0, z_0). \tag{17}$$

Comparing (17) with the results of the linear theory, we see that deviations from linear background distributions can be described by formulas similar to (12) in which the left- and right-hand sides are expressed in terms of Eulerian and Lagrangian coordinates, respectively. In the linear theory, $x_0 \sim x$ and $z_0 \sim z$; i.e., formulas (12) and (17) are equivalent. In the nonlinear problem, the relation between Lagrangian and Eulerian coordinates can be used to describe some intricate distributions evolving from smooth initial conditions. The most striking result of this evolution is the formation of discontinuities (jumps) in vertical distributions.

In what follows, we consider initial distributions of the form

$$T_i = \Delta T h\left(\frac{x}{L}\right)\tau\left(\frac{z}{H}\right),$$

where ΔT, L, and H are the amplitude and the horizontal and vertical length scales of T_i, respectively. The even nonnegative function $h(x)$ vanishes at infinity and satisfies the condition $h(0) = 1$. Using (14) and (15), we obtain the following relations between the dimensionless Eulerian and Lagrangian coordinates (appropriately normalized to length scales L and H):

$$z = z_0 + ah(x_0)\tau(z_0),$$
$$x = x_0 - a\int_0^{x_0}\frac{h(x_0)\tau'(z_0)}{1 + ah(x_0)\tau'(z_0)}dx_0, \tag{18}$$

where $a = \alpha\Delta T/\gamma H$ and the prime denotes a derivative. Note that the expression for x in (18) is derived by integrating the equation $\partial x/\partial x_0 = (\partial z/\partial z_0)^{-1}$ under a symmetry condition, $x = 0$ for $x_0 = 0$. The function $z_0 = z_0(x_0, z)$ in the integrand is determined from the expression for z.

The relations between coordinates can be found in explicit form in the following simple example. Consider the fluid layer $0 < z < 1$. Suppose that $\tau(z_0)$ is a piecewise linear function:

$$\tau(z_0) = \begin{cases} z_0, & 0 < z_0 < 0.5, \\ 1 - z_0, & 0.5 < z_0 < 1 \end{cases}$$

(the temperature disturbance has a maximum on the centerline of the layer and vanishes on its boundaries). Then, relations (18) reduce to

$$z = z_0(1 + ah(x_0)),$$
$$x = x_0 - a\int_0^{x_0}\frac{h(x_0)}{1 + ah(x_0)}dx_0, \quad 0 < z_0 < 0.5,$$
$$z = z_0 + ah(x_0)(1 - z_0), \tag{19}$$
$$x = x_0 + a\int_0^{x_0}\frac{h(x_0)}{1 - ah(x_0)}dx_0, \quad 0.5 < z_0 < 1.$$

When $a > 0$, formulas (19) describe the displacements of fluid elements due to upward convective flow and spreading near the upper boundary. Using (19) to express z_0 in terms of z and substituting the result into (17), we find that the deviation from the linear background temperature distribution is

$$T'_f = \Delta T \frac{\eta}{\eta - 1} h(x_0) \times \begin{cases} z/(1+a), & 0 < z < 0.5(1+a), \\ (1-z)/(1-a), & 0.5(1+a) < z < 1. \end{cases} \quad (20)$$

Figure 1 shows the temperature profiles at $x_0 = x = 0$ corresponding to several values of a. The figure illustrates a tendency toward formation of discontinuities in the final vertical distributions.

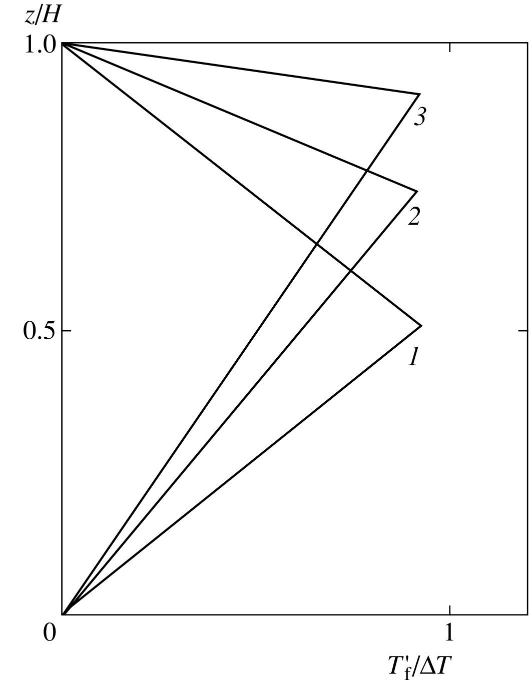

Fig. 1. Temperature deviation from background linear distribution for several values of parameter a in the case of piecewise linear $\tau(z_0)$: $\eta = 2$; $a \ll 1$ (*1*), $a = 0.5$ (*2*), 0.8 (*3*).

Next, we analyze the structure of vertical temperature distributions developing from arbitrary smooth profiles $\tau(z_0)$. According to (16) and (18), the distribution $T_f = T_f(z)$ on the symmetry axis $x_0 = 0$ of the temperature disturbance is described parametrically by the expressions

$$z = z_0 + a\tau(z_0), \quad T_f = \gamma_T H[z_0 + a(1-\eta)\tau(z_0)], \quad (21)$$

which are obtained by using the relation $\Delta T/\gamma_T H = a(1-\eta)$. Figure 2a shows distributions (21) in a vertically unbounded fluid developing from a smooth localized distribution $\tau(z_0) > 0$ for several values of a and $\eta < 0$. The figure demonstrates that a discontinuity appears in the vertical T_f profile at a point $z = z_{cr}$ as the parameter a reaches a certain value $a = a_{cr}$. A similar discontinuity develops in the vertical salinity profile (see Fig. 2b). These discontinuities cancel out, so that the vertical density distribution retains a smooth (linear) profile.

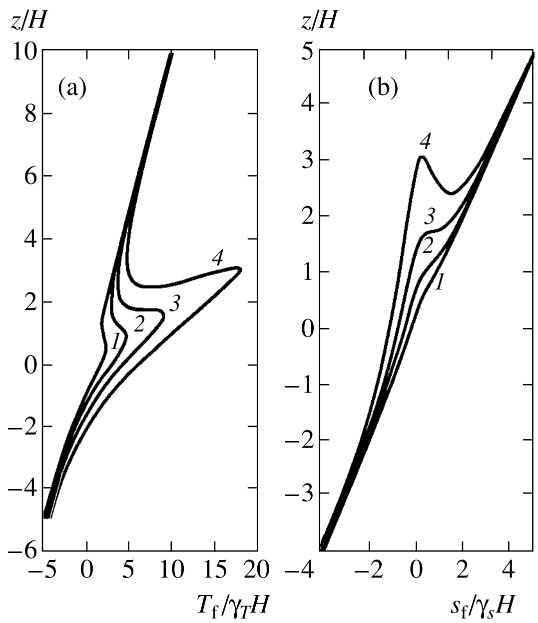

Fig. 2. Vertical (a) temperature and (b) salinity distributions on the symmetry axis of disturbance with $\tau(z_0) = 1/(1 + z_0^2)$ and $\eta = -5$ for $a = 0.4$ (*1*), 0.8 (*2*), 3 (*4*), and $a = a_{cr} = 8\sqrt{3}/9 \approx 1.53$ (*3*).

Now, we estimate the critical values of parameters. By following [12], it can be shown that the formation of a discontinuity corresponds to the appearance of an inflection point z_* in the curve of $z = z(z_0)$, which can be calculated by solving the equation $\tau''(z_0) = 0$. Since

$$\frac{\partial T}{\partial z} = \frac{\partial T/\partial z_0}{1 + a\tau'(z_0)},$$

we obtain

$$a_{cr} = -\frac{1}{\tau'(z_*)}, \quad z_{cr} = z_* + a_{cr}\tau(z_*).$$

In particular, if $\tau(z_0) = 1/(1 + z_0^2)$, then $a_{cr} = 8\sqrt{3}/9 \approx 1.54$ and $z_{cr} = \sqrt{3}$. If $\tau(z_0) = \exp(-z_0^2)$, then $a_{cr} = \exp(0.5)/\sqrt{2} \approx 1.17$ and $z_{cr} = \sqrt{2}$. The final vertical temperature and salinity distributions are smooth when $a < a_{cr}$ and many-valued (discontinuous) when $a > a_{cr}$. Using these parameter values, we find that a discontinuity appears as ΔT exceeds $\Delta T = a_{cr}\gamma H/\alpha$; i.e., the critical

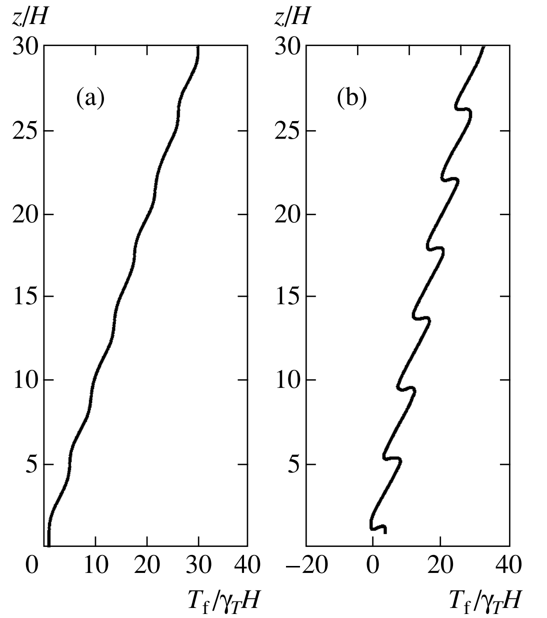

Fig. 3. Vertical temperature distribution on the symmetry axis of disturbance with $\tau(z_0) = \cos(\lambda z_0)$ for $\lambda = 1.5$, $\eta = -2$, and $a = 0.2$ (a) and $a = 0.95 > a_{cr}$ (b).

value is a linear function of the scale H. Using the parameter values $H = 10$ m, $\alpha = 2 \times 10^{-4}$ K^{-1}, and $\gamma = 9 \times 10^{-7}$ m^{-1} ($N = 3 \times 10^{-3}$ s^{-1}) characteristic of the ocean, we obtain $\Delta T = 7 \times 10^{-2}$ K. Thus, even an initial temperature disturbance with an amplitude of about a tenth of kelvin develops into a discontinuous profile.

It is important that the formation of discontinuities is a universal effect independent of details of the profile $\tau(z_0)$. This effect has a simple physical explanation. Indeed, according to (14), the vertical displacement $l = z - z_0$ is determined by the amplitude ΔT of the initial disturbance T_i. When $\Delta T > 0$, relatively warm fluid parcels move upwards faster than cooler ones. This leads to the formation of a discontinuity analogous to the well-known N-wave in gas dynamics [13]. Profiles of this kind are frequently measured in the ocean, being characteristic of temperature inversions [7, 8].

Figure 3 shows the distributions predicted by (21) for $\tau(z_0) = \cos(\lambda z_0)$ and two values of a. The figure demonstrates that a periodic vertical distribution generated, for example, by an internal wave can develop into a steady sawtooth temperature profile consisting of alternating regions of constant gradient and jumps (inversions). The jump formation condition for a periodic disturbance is $a > a_{cr} = 1/\lambda$. The critical value of ΔT corresponding to $\lambda = 1$ and parameter values given above is 4.5×10^{-2} K. In the case of an aperiodic initial distribution with regions of positive and negative vertical gradients, an irregular profile will obviously develop. Profiles of this kind are characteristic of oceanic fine structure, which consists of "irregular or systematic vertically alternating intervals of low and high vertical gradients of some property" [7]. Note that the temperature jump characteristic of this fine structure is several tenths of kelvin, which corresponds to the supercritical regime.

5. INTEGRAL AND LAGRANGIAN CONSERVATION LAWS FOR THE SYSTEM OF FLUID-DYNAMICS EQUATIONS FOR ADIABATIC FLOWS

Hydrodynamic "memory" of stratified two-component fluids is associated with some specific integral and Lagrangian conservation laws. Some of these laws are discussed in this section.

First, we explain the hydrodynamic mechanism that underlies local conservation law (9), which is responsible for "trace" formation in the linear theory. It was mentioned above that the variables T and s are Lagrangian invariants in the nonlinear problem since they are conserved in each fluid element. Therefore, the lines of intersection of surfaces $T = \text{const}$ and $s = \text{const}$ are material lines. By virtue of (3), the vector field $\mathbf{R}(\mathbf{x}, t)$ tangent to these lines,

$$\mathbf{R}(\mathbf{x}, t) = \nabla T \times \nabla s, \tag{22}$$

satisfies the equation $\partial\mathbf{R}/\partial t = \text{curl}[\mathbf{u} \times \mathbf{R}]$, which can be rewritten by using a well-known vector identity as

$$\frac{d\mathbf{R}}{dt} = (\mathbf{R} \cdot \nabla)\mathbf{u}. \tag{23}$$

Equation (23) is known as the "freezing-in" equation [14]. When (23) holds, the $\mathbf{R}$ field lines are material lines, and the vector-tube intensities are conserved.

Conservation law (9) is equivalent to the linearized form of Eq. (23). Indeed, substituting $T = \bar{T} + T'$ and $s = \bar{s} + s'$ into (22), we obtain

$$\mathbf{R} = (\gamma_T \mathbf{k} + \nabla T') \times (\gamma_s \mathbf{k} + \nabla s') \sim \mathbf{R}' = \nabla r \times \mathbf{k}$$

for small deviations. In the linear approximation, Eq. (23) reduces to $\partial\mathbf{R}'/\partial t = 0$, which is equivalent to (9) for disturbances bounded at infinity.

Another Lagrangian invariant of system (1)–(3) (independent of T and s) is the Ertel potential vorticity [14]

$$dq/dt = 0, \quad q = \text{curl}\mathbf{u} \cdot \nabla\rho.$$

It is obvious that Eq. (23) is satisfied for any field $\mathbf{R}$ defined by two of the three invariants T, s, and q, such as $\mathbf{R} = \nabla\rho \times \nabla q$.

"Freezing-in" equation (23) is associated with interesting conservation laws. It can easily be shown that vector fields **u** and **R** satisfying, respectively, system (1)–(3) and Eq. (23) satisfy the following equation as well:

$$\frac{d}{dt}(\mathbf{R}\cdot\mathbf{u}) = \operatorname{div}(L\mathbf{R}) + p\mathbf{R}\cdot\nabla\left(\frac{1}{\rho}\right), \tag{24}$$

where

$$L = \frac{\mathbf{u}^2}{2} - \frac{p}{\rho} - gz.$$

If $\mathbf{R} = \nabla\rho\times\nabla q$, then the second term on the right-hand side in (24) vanishes. Therefore, the integral of this equation over a volume V yields the conservation law

$$\frac{\partial I_1}{\partial t} = 0, \quad I_1 = \int_V [\nabla\rho\times\nabla q]\cdot\mathbf{u}d\mathbf{x}, \tag{25}$$

if the boundary of V consists of segments that are tangent to **R**.

Analogously, substituting $\mathbf{R} = \nabla T \times \nabla s$ into Eq. (24), we obtain the conservation law

$$\frac{\partial I_2}{\partial t} = 0, \quad I_2 = \int_V [\nabla T\times\nabla s]\cdot\mathbf{u}d\mathbf{x}. \tag{26}$$

Conservation laws (25) and (26) hold if the integrands are multiplied by an arbitrary function $\mu(T, s, q)$, because the field $\mu\mathbf{R}$ satisfies (23).

A conservation law analogous to (25) was found for a general compressible flow of a one-component fluid in [15]. Conservation law (26), which is valid specifically for two-component fluids, was obtained in [16] by using a Hamiltonian formalism (Casimir C_3). Conservation laws (25) and (26) are analogous to the Moffatt and Volterra conservation laws in Newtonian fluid dynamics and magnetohydrodynamics [14, 15], being independent of the energy conservation law

$$\frac{\partial E}{\partial t} = 0, \quad E = \int_V \left[\frac{\mathbf{u}^2}{2} + gz\right]\rho d\mathbf{x}.$$

Note also that adiabatic flows of two-component fluids have another property related to the Cauchy problem for system (1)–(3). Indeed, this problem can be solved by reducing Eqs. (3) to a single equation for density and finding the fields **u**, p, and ρ from the "reduced" system

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p - g\mathbf{k}, \quad \frac{d\rho}{dt} = 0, \quad \operatorname{div}\mathbf{u} = 0, \tag{27}$$

which is equivalent to the equations of motion for a one-component incompressible fluid. Once these fields are found, the temperature and salinity fields are determined by using the known density and velocity fields and the initial conditions. Thus, convection in a two-component fluid system can be described by computing Eqs. (27) and calculating the temperature and salinity fields at each time step.

6. CONCLUSIONS

Scenarios of hydrostatic adjustment in a stratified two-component fluid system differ substantially from those predicted by standard models of temperature-stratified fluids. Binary fluid mixtures have "memory": horizontally nonuniform disturbances in the initial temperature and salinity distributions do not vanish at the final stage of the process, transforming into a persistent "trace." Moreover, both amplitude and structure of the "trace" exhibit unexpected behavior. For example, depending on the relative contributions of the two fields to background stratification, the amplitude of deviation from the background temperature field in the final state can be much larger than the initial one and have opposite sign. In a certain sense, doubly unstable fluid systems of the type discussed here can be considered as active media.

One universal feature of the "trace" structure is the development of discontinuous vertical distributions (thermohaline staircases) from smooth initial conditions. Distributions of this kind are characteristic of oceanic fine structure. At the final stage of hydrostatic adjustment, the temperature and salinity profiles typically compensate each other in the density field. For example, a smooth density distribution in a hydrostatically balanced stratified ocean may be the result of mutual compensation of highly nonuniform temperature and salinity profiles. Such distributions have been found by analyzing the results of numerous oceanographic observations [5–8]. The phenomena discussed in this paper should be examined in special experimental studies.

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APPENDIX

Determination of Temperature and Salinity Fields from Known Density and Velocity Distributions and Initial Conditions

Consider the Cauchy problem for system (1)–(3) in the layer $0 \le z \le H$, subject to the impermeability condition $w = 0$ on the horizontal boundaries $z = 0$ and $z = H$. In what follows, we show that the fields T and s can be uniquely determined at any instant by specifying initial conditions and using known density and velocity fields (which are found by solving system (27)). In the three-dimensional problem, the temperature distribution can be found by solving the first-order quasilinear partial differential equation

$$\frac{\partial(T, \rho, q)}{\partial(x, y, z)} = F(T, \rho, q), \quad \text{(A.1)}$$

subject to the boundary conditions

$$T|_{z=0} = \varphi_0(\rho, q), \quad T|_{z=H} = \varphi_H(\rho, q). \quad \text{(A.2)}$$

Here, the functions F, φ_0, and φ_H are determined by the initial conditions and q is the Ertel potential vorticity.

For two-dimensional motions in the xz plane, the temperature distribution is uniquely determined if the density field is known by solving the boundary value problem

$$\frac{\partial(T, \rho)}{\partial(x, z)} = F(T, \rho),$$
$$T|_{z=0} = \varphi_0(\rho), \quad T|_{z=H} = \varphi_H(\rho). \quad \text{(A.3)}$$

The statement of problem (A.1), (A.2) is substantiated as follows. In the three-dimensional problem, the invariants T, s, and q can be used to construct an infinite set of invariants by applying Ertel's commutation theorem [15]:

$$\frac{d}{dt}[f, g, h] = [\dot{f}, g, h] + [f, \dot{g}, h] + [f, g, \dot{h}]. \quad \text{(A.4)}$$

Here, the bracket denotes a Jacobian with respect to x, y, and z and the upper dot denotes a total derivative. Substituting T, ρ, and q into (A.4), we obtain the Lagrangian conservation law

$$\frac{dJ}{dt} = 0, \quad J = \frac{\partial(T, \rho, q)}{\partial(x, y, z)}, \quad \text{(A.5)}$$

which means that the fluid volume bounded by surfaces of constant T, ρ, and q remains invariant.

Let us show that, if the initial distributions T_0, ρ_0, and q_0 are functionally independent, then the invariant J remains a single-valued function of T, ρ, and q in the course of evolution.

Indeed, the definition of Lagrangian invariant implies that $T = T_0(\mathbf{x}_0)$, $\rho = \rho_0(\mathbf{x}_0)$, $q = q_0(\mathbf{x}_0)$, and $J = J_0(\mathbf{x}_0)$ at any $t > 0$, where $\mathbf{x}_0 = (x_0, y_0, z_0)$ are Lagrangian coordinates. Using the first three relations to express $\mathbf{x}_0$ in terms of T, ρ, and q and substituting the result into the last one, we obtain a functional relation $J = F(T, \rho, q)$ equivalent to Eq. (A.1). The existence of this relation is due to the following mathematical fact: "For any vector field $\mathbf{u}(\mathbf{x}, t)$ in the n-dimensional space $R^n_{\mathbf{x}}$, there exist exactly n functionally independent (basis) Lagrangian invariants such that any invariant can be expressed in terms of these ones" [17].

Moreover, this proposition entails boundary conditions (A.2). Indeed, since the velocity field is two-dimensional on the boundaries ($w = 0$ if $z = 0$ or $z = H$), the invariant T can be expressed in terms of the basis invariants ρ and q. The functions contained in (A.2) are determined by eliminating x and y from the initial conditions: at $z = 0$, we have $T = T_0(x, y, 0)$, $\rho = \rho_0(x, y, 0)$, and $q = q_0(x, y, 0)$; analogous relations hold at $z = H$.

Once T is found, the salinity distribution can be determined either by using an equation of state $\rho = \rho(T, s)$ or by solving a boundary value problem analogous to (A.1), (A.2).

Boundary value problem (A.3) is formulated analogously. In the two-dimensional case, equations $d\rho/dt = 0$ and $dT/dt = 0$ are combined to obtain a Lagrangian conservation law [12]:

$$\frac{dJ}{dt} = 0, \quad J = \frac{\partial(T, \rho)}{\partial(x, z)}. \quad \text{(A.6)}$$

Again, it can readily be shown that $J = F(\rho, T)$ at any instant and the invariant T on the boundaries can be uniquely expressed in terms of ρ by using initial conditions. Moreover, by substituting $\rho = \bar{\rho}(z)$ into (A.3), we can find the final temperature distribution that forms as a result of hydrostatic adjustment. However, the Lagrangian approach used in Section 5 is more constructive.

Note also that the distributions T' and s' at any instant are uniquely determined by the buoyancy field $\sigma = \alpha T' - \beta s'$ and the initial conditions. Indeed, conservation law (9) implies that

$$\gamma_s T' - \gamma_T s' = \gamma_s T_i - \gamma_T s_i$$

at any t. This relation combined with the expression for σ can be considered as a system of linear equations for T' and s'. Solving this system, we obtain

$$T' = \frac{\alpha\eta T_i - \beta s_i - \sigma}{\alpha(\eta - 1)}, \quad s' = \frac{\alpha\eta T_i - \beta s_i - \eta\sigma}{\beta(\eta - 1)}. \quad \text{(A.7)}$$

When $\sigma = 0$, these expressions reduce to (11). The distribution of σ at any instant is found by solving Eq. (8).

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