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Nontrivial features in the hydrodynamics of seawater and other stratified solutions

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Contents

1. Introduction	**356**
2. Convective instability driven by double diffusion	**357**
3. Hydrostatic adjustment in a 'doubly-stratified' fluid. Hydrodynamic 'memory'	**357**
3.1 The phenomenon of hydrodynamic 'memory'; 3.2 Problem formulation; 3.3 Linear approximation; 3.4 Nonlinear theory; 3.5 Integral and Lagrangian conservation laws for the system of equations of adiabatic dynamics	
4. Formation of compensated inhomogeneities in shear flows	**364**
5. Anomalous responses to mechanical and thermal surface forcing	**365**
5.1 The problem of inhomogeneous tangent stresses and preliminary estimates; 5.2 The solution and its analysis; 5.3 Responses to thermal forcing	
6. New mechanisms of convective instability	**369**
6.1 A new type of instability caused by double diffusion; 6.2 Anomalous thermocapillary instability in two-component media; 6.3 Convective instability caused by a background flow; 6.4 Instability caused by a faster transport of one of the components; 6.5 Convective–radiative instability of moist air; 6.6 Instability linked to phase transitions at a boundary	
7. Effects of rotation	**376**
8. Density currents caused by double diffusion	**378**
9. Anomalous hydrodynamic drag	**378**
10. Convection: nonlinear interaction between fields of two tracers	**380**
11. Conclusions	**380**
References	**380**

Abstract. **Stratified two-component media (for example, sea water) can have general hydrothermodynamic properties widely different from the properties of 'usual' fluids, whose density depends on temperature only. For example, temperature perturbations in such media can increase despite a hydrostatically stable density stratification. In this review, a number of recently discovered physical mechanisms and phenomena are discussed, including the mechanisms of convective instability, the hydrodynamic 'memory' of two-component media, the formation of temperature and admixture concentration jumps, the anomalous response of binary mixtures to mechanical and thermal forcing, and the effective 'negative heat capacity'.**

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1. Introduction

Several decades ago, a number of surprising, in the words of J Turner [1], phenomena were first discovered in stratified two-component fluids (such as seawater) subject to the action of gravitational force. For example, it turned out that a convective instability can develop in such fluids even if their density stratification is hydrostatically stable (the background density decreases with height). Since then, phenomena of this kind, being of large scientific and practical interest, have been the subject of active research (see, e.g., Refs [1–9]). Review [2] alone, written nearly 30 years ago, already mentions more than one hundred publications. Their number has increased immensely to the present time.

This review cannot pretend to cover all this extensive material. Its scope does not embrace the important field of research dealing with the formulation and analysis of amplitude equations for nonlinear supercritical convection modes (the generalized Ginzburg–Landau equations; see, e.g., Refs [10–15] and the references therein). Likewise, the results of laboratory and numerical modeling of convection in binary mixtures are typically not considered, either. The review is limited to the description of certain classes of physical mechanisms and phenomena discovered (mainly by us) over the last 10–15 years. They include new mechanisms of convective instability, the phenomenon of hydrodynamical 'memory' in two-component media, anomalous responses of binary mixtures to mechanical and thermal forcing, an effective 'negative heat capacity' of such media, possible formation of temperature jumps, and anomalous hydrodynamic drag.

2. Convective instability driven by double diffusion

The phenomenon of convective instability caused by double diffusion was discovered already rather long ago and is described in detail in many publications (see, e.g., Refs [1, 2, 5, 6, 8, 9]). For convenience, we briefly recall the main physical mechanism.

For definiteness, we generally take seawater as an example of a binary mixture; its equation of state can be approximated with an accuracy sufficient for our purposes by the linear equation [1]

$$\rho = \rho_*(1 - \alpha T + \beta S)\,, \tag{1}$$

where ρ_* is the reference density ρ at constant mean temperature T_* and salinity S_*, T and S are the respective deviations from the mean values, and α and β are respectively the thermal expansion and saline contraction coefficients. If the background distributions of temperature and salinity, $\bar{T}$ and $\bar{S}$, depend only on the coordinate z (the z axis is directed vertically upward), the medium is in the state of mechanical (hydrostatic, or simply static) equilibrium. In this section (and in most cases below), we suppose that the vertical gradients of these background distributions, respectively denoted as γ_T and γ_S, are uniform. We also assume that the background density stratification is statically stable: the density decreases with height (yet the contribution to density stratification of one of the components — the temperature or salinity — can be unstable).

We suppose that at the initial time instant, a perturbation is imposed in the form of a small vertical (for definiteness, upward) displacement of a certain fluid volume. Because the stratification was assumed to be stable, this volume in its new position is denser and heavier than the ambient fluid, and acquires a negative buoyancy. Therefore, at first glance, the appearance of returning force seems unavoidable. But the buoyancy of the volume also depends on the processes of exchange with its surroundings. A detailed consideration shows that the last circumstance can in general change the sign of the emerging buoyancy forces. This is possible if the two substances (heat and salt) contribute to the background density stratification with different signs and diffuse at different rates (for instance, the diffusivity of salt in seawater is approximately 100 times less than that of heat).

For example, let the fluid be stably stratified in temperature (its temperature increases upward). Let the salinity stratification be unstable but contributing less strongly to the background density, such that the latter decreases with height. A fluid parcel displaced upward, at first sight, must be colder and fresher, and on the whole denser than the ambient medium. But the heat exchange with the ambient medium proceeds much faster than the exchange in salt. As a result, the deviation of temperature in the displaced parcel can relax rapidly (depending on the spatial scales of perturbation), in contrast to the salinity perturbation, which contributes to the density with the opposite sign. Thus, generally speaking, situations can arise when the displaced volume is lighter, not denser, than the ambient medium. This implies the existence of a positive feedback and the development of a perturbation.

This is, in a nutshell, the essence of the mechanism of convective instability caused by the difference in diffusivities and called double-diffusive convection. This effect has been studied both theoretically and experimentally. It turned out to be important not only in oceanography but also in a number of other applications [1, 2, 4–6].

We briefly discuss the opposite case where a stable density stratification is set by salinity, while the temperature contribution is destabilizing, remaining smaller in absolute value. Using the same line of reasoning, we arrive at the conclusion that in this case, the difference in the diffusivities ('differential' diffusion) should lead not to the disappearance but to the amplification of the returning force applied to the vertically displaced fluid parcel. It is known, however, that in some cases, the emergence of large returning forces, instead of stabilizing the system, can excite its oscillations and result in the occurrence of an oscillatory instability (overstability). And indeed, linear analysis has revealed that a subdomain exists in the parameter space of two-component media where the oscillatory instability can develop [1, 2].

3. Hydrostatic adjustment in a 'doubly-stratified' fluid. Hydrodynamic 'memory'

3.1 The phenomenon of hydrodynamic 'memory'

In this section, we show that the process of mechanical equilibration in ideal stratified two-component media can proceed differently from that in usual fluids stratified only in temperature [16, 17]. In the latter case, as is well known [18], initial temperature inhomogeneities in the field of gravitational force decay with time owing to wave motions generated in the presence of such inhomogeneities, which 'level off' the inhomogeneities and fade away in space. This is essential, for example, for the dynamics of geophysical fluids (the atmosphere, ocean, or liquid interiors of planets). In particular, it greatly complicates the explanation of the existence and abundance of certain types of 'long-lived' thermal inhomogeneities in the upper layers of seawater (see, e.g., Refs [19, 20]). From the analysis in Sections 3.2–3.5, it follows that in two-component media, the initial inhomogeneities in the temperature and admixture concentration distributions do not disappear even in the final stages of mechanical equilibration. In other words, the two-component media preserve the 'memory' of initial perturbations, allowing the generation and prolonged existence of horizontally inhomogeneous 'traces' in temperature and admixture concentration distributions. The emergence of long-lived thermohaline traces (related to temperature and salinity) was reported long ago not only in observations of the ocean upper layer but also in laboratory experiments (see, e.g., Ref. [21, p. 221]); however, a reasonably rigorous explanation of phenomena of that kind had been lacking until recently.

The universal feature of the structure of 'traces' is the presence of discontinuities (jumps) that can evolve from smooth initial conditions. Structures of this type (especially jumps in vertical distributions, or 'staircases') are abundant in the upper layer of the ocean [22, 23], and the explanation of their origin presents a pressing problem.

From a hydrodynamic standpoint, the aspect of memory of two-component media, discovered in Refs [16, 17], is linked to the fact that for adiabatic motions, the temperature and admixture concentration are Lagrangian invariants, i.e., the quantities preserved in fluid particles. This implies a set of specific integral and Lagrangian conservation laws, which are also discussed below.

It is considered obvious that in a medium with stable density stratification, hydrodynamic perturbations would decay because vertical displacements are opposed by returning forces. This makes the result in Section 3.3 even more intricate: in binary mixtures, perturbations in temperature and admixture concentration (salinity) may amplify by a large factor for stable stratifications, arbitrarily strong (even if the effects of double diffusion mentioned in Section 2 are excluded). In this sense, such 'doubly nonequilibrium' fluids (stratified in both components) behave as active media.

3.2 Problem formulation

We consider the behavior of perturbations in an ideal quiescent unbounded two-component fluid in a gravitational field. For definiteness, here (and in most cases below), we mean saline water stratified in both temperature and admixture concentration (salinity), such that its hydrostatic equilibrium is stable. Taken separately, the thermal and saline contributions to the density stratification can be unstable, but the net density stratification is stable.

According to the traditionally used approximation [1, 3, 4, 22], we use linear equation of state (1). With Eqn (1), adiabatic motions of the fluid are described by the closed system of equations of hydrodynamics and admixture concentration transport

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{1}{\rho}\,\nabla p - g\mathbf{k}\,, \qquad \operatorname{div}\mathbf{u} = 0\,, \tag{2}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 0\,, \qquad \frac{\mathrm{d}S}{\mathrm{d}t} = 0\,, \tag{3}$$

where $\mathbf{u}$ is the velocity vector with the respective components u, v, and w along the horizontal axes x and y and the z axis directed upward, p is the pressure, g is the acceleration of gravity, $\mathbf{k}$ is a unit vertical vector, and $\mathrm{d}/\mathrm{d}t = \partial/\partial t + (\mathbf{u}, \nabla)$ is the operator of the full derivative.

The initial conditions for Eqns (1)–(3) are written as

$$\mathbf{u}\Big|_{t=0} = \mathbf{0}\,, \quad T\Big|_{t=0} = \gamma_T z + T_{\mathrm{i}}(\mathbf{x})\,, \quad S\Big|_{t=0} = \gamma_S\, z + S_{\mathrm{i}}(\mathbf{x})\,, \tag{4}$$

where $\mathbf{x} = (x, y, z)$ and T_{i} and S_{i} are the given initial perturbations. The uniform background gradients of each of the components, γ_T and γ_S, as mentioned, are selected such that the background state is statically stable, i.e., the background density $\bar{\rho}(z) = \rho_*(1 - \gamma z)$ decreases with height. In a stable state, $\gamma = \alpha\gamma_T - \beta\gamma_S > 0$.

The relative contributions of temperature and admixture concentration to the background density stratification can be conveniently characterized by the dimensionless parameter

$$\eta = \frac{\beta\gamma_S}{\alpha\gamma_T}\,, \tag{5}$$

whence $\gamma = \alpha\gamma_T(1 - \eta)$. One-component temperature-stratified fluids, considered most frequently, correspond to $\eta = 0$. In some cases, it is more convenient to use the parameter

$$m = \frac{\eta}{\eta - 1} = \frac{\beta\gamma_S}{\beta\gamma_S - \alpha\gamma_T} = \frac{\mathrm{d}\bar{\rho}_S/\mathrm{d}z}{\mathrm{d}(\bar{\rho}_S + \bar{\rho}_T)/\mathrm{d}z}\,, \tag{5a}$$

where $\mathrm{d}\bar{\rho}_T/\mathrm{d}z$ and $\mathrm{d}\bar{\rho}_S/\mathrm{d}z$ are the respective background vertical density gradients due to temperature and admixture concentration. Accordingly, the parameter m acquires the meaning of the relative contribution of the admixture to the background density stratification.

In an unbounded stably stratified medium, the initial horizontal inhomogeneities of the density field $\rho_{\mathrm{i}} = -\rho_*(\alpha T_{\mathrm{i}} - \beta S_{\mathrm{i}})$ result in wave motions that smooth out the inhomogeneities in density and decay as time progresses. The accompanying process of hydrostatic adjustment can be analyzed most simply in the case of small-amplitude perturbations.

3.3 Linear approximation

Using primes to denote small perturbations of thermodynamic variables and using the Boussinesq approximation, instead of Eqns (1)–(3), we obtain the linear system of equations

$$\frac{\partial\mathbf{u}}{\partial t} = -\frac{1}{\rho_*}\,\nabla p' + g(\alpha T' - \beta S')\,\mathbf{k}\,, \qquad \operatorname{div}\mathbf{u} = 0\,, \tag{6}$$

$$\frac{\partial T'}{\partial t} + \gamma_T w = 0\,, \qquad \frac{\partial S'}{\partial t} + \gamma_S w = 0\,, \tag{7}$$

with the initial conditions $\mathbf{u}\big|_{t=0} = \mathbf{0}$, $T'\big|_{t=0} = T_{\mathrm{i}}(\mathbf{x})$ and $S'\big|_{t=0} = S_{\mathrm{i}}(\mathbf{x})$. For the dimensionless buoyancy

$$\sigma = -\frac{\rho'}{\rho_*} = \alpha T' - \beta S'\,,$$

system of equations (6) and (7) reduces to the single equation

$$\frac{\partial^2}{\partial t^2}\,\Delta_3\sigma + N^2\Delta_2\sigma = 0\,, \tag{8}$$

where $N = \sqrt{g\gamma}$ is the Brunt–Väisälä frequency (the buoyancy frequency), and Δ_3 and Δ_2 are three- and two-dimensional Laplace operators (with respect to spatial coordinates). Equation (8) is the main equation of the linear theory of internal gravity waves [24] (it is usually formulated for the vertical velocity). The general solution of the Cauchy problem for this equation is given in Ref. [25]. From the solution, describing the process of wave propagation in space, it follows that density perturbations decay with time, $\sigma \to 0$ as $t \to \infty$.

In a one-component medium (whose density is only temperature dependent), the temperature perturbation T', obviously, decays together with σ. The behavior of T' is totally different in a two-component medium. We let T_{f} and S_{f} denote final (at $t \to \infty$) perturbations of the temperature and admixture concentration. To find them, we use a simple local conservation law that is a linearized form of the evolution equation for frozen-in fields (see Section 3.5), which is specific for two-component media. This law is obtained by eliminating w from Eqns (7):

$$\frac{\partial r}{\partial t} = 0\,, \qquad r = \gamma_S T' - \gamma_T S'\,. \tag{9}$$

According to Eqns (9), the field $r(\mathbf{x})$ does not change in time,

$$\gamma_S T_{\mathrm{f}} - \gamma_T S_{\mathrm{f}} = \gamma_S T_{\mathrm{i}} - \gamma_T S_{\mathrm{i}}\,. \tag{10}$$

Additionally, $\sigma = 0$ in the final state, i.e., $\alpha T_{\rm f} = \beta S_{\rm f}$. From the last relation and Eqn (10), we obtain the expressions

$$T_{\rm f} = \frac{\alpha\eta T_{\rm i} - \beta S_{\rm i}}{\alpha(\eta - 1)}\,, \qquad S_{\rm f} = \frac{\alpha\eta T_{\rm i} - \beta S_{\rm i}}{\beta(\eta - 1)}\,, \tag{11}$$

which demonstrate that in a two-component medium, the perturbations T' and S' also persist through the final stage of the hydrostatic adjustment process. These perturbations compensate each other in the density field ($\sigma_{\rm f} = 0$), forming a stationary trace, which can be called thermohaline for salt water. We note that in real two-component fluids, the traces fade out for the characteristic dissipation time, although this time considerably exceeds the time scale of hydrostatic adjustment for perturbations with not too small spatial scales.

Relations (11) give only the final distributions of temperature and salinity. Generally, by determining the fields of buoyancy σ and invariant r from Eqns (8) and (9), we find these distributions at an arbitrary time instant in the form

$$T' = \gamma^{-1}(\gamma_T\sigma - \beta r)\,, \qquad S' = \gamma^{-1}(\gamma_S\sigma - \alpha r)\,. \tag{12}$$

Relations (12) directly follow from the expressions for σ and r, interpreted as a system of linear equations for T' and S'. According to these relations, the thermohaline field (T', S') (the field of temperature and salinity distributions) can be expressed as a sum of 'density-related' ('wave') (contributing to the density perturbations and, consequently, influencing the fluid dynamics) and 'compensated' components,

$$\begin{gathered}(T', S') = (T_\sigma, S_\sigma) + (T_r, S_r)\,, \\ (T_\sigma, S_\sigma) = \gamma^{-1}\sigma(\gamma_T, \gamma_S)\,, \qquad (T_r, S_r) = -\gamma^{-1}r(\beta, \alpha)\,.\end{gathered} \tag{13}$$

Distinct from the density-related component, directly participating in the wave dynamics, the compensated component (T_r, S_r) is preserved in the process of evolution and does not contribute to the buoyancy field (it behaves as a passive conservative tracer), $\alpha T_r = \beta S_r$. We emphasize that decomposition (13) is unique and bears a rather general character.

We explore specific features of final distributions in the case $S_{\rm i} = 0$ (the initial perturbation in the admixture concentration is absent):

$$T_{\rm f} = \frac{\eta}{\eta - 1}\,T_{\rm i}\,, \qquad S_{\rm f} = \frac{\alpha}{\beta}\,\frac{\eta}{\eta - 1}\,T_{\rm i}\,. \tag{14}$$

For $\eta = 0$ (a one-component medium), the trace of the initial perturbation vanishes: $T_{\rm f} = S_{\rm f} = 0$. For $\eta \neq 0$, with the constraint $\gamma > 0$, three qualitatively different situations are possible, depending on the value of the parameter $m = \eta/(\eta - 1)$.

(1) *The background stratifications of temperature and admixture are stable*: $\gamma_T > 0$, $\gamma_S < 0$, $\eta < 0$. In this case, $0 < m < 1$, i.e., the final temperature perturbation preserves the sign of the initial one, but loses amplitude. While the reduction in amplitude looks quite natural for a stably stratified medium, it is nontrivial that the perturbation does not decay altogether.

(2) *The temperature stratification is unstable, but the system is stabilized due to the contribution of the admixture*: $\beta\gamma_S < \alpha\gamma_T < 0$, $\eta > 1$. In this case, $m > 1$, i.e., the amplitude of perturbation in the final state always exceeds the initial one, and in the limit $\eta \to 1 + 0$ (the limit of a neutral density stratification), $T_{\rm f} \to \infty$ ($m \to \infty$). The effect of perturbation amplification in a stably stratified two-component fluid can be explained as follows. An initial positive temperature perturbation leads to ascending motions in the fluid, which for $\gamma_T < 0$ carry warmer fluid parcels from below. If the density stratification is close to neutral, it does not counteract the rapid development of such motions. This positive feedback leads to the formation of an intense trace. The dimensionless parameter m has the meaning of the amplification coefficient of the initial perturbation. We note that the amplification is in principle possible for any stable density stratification. This can be readily seen from expression (5a), where the denominator $\mathrm{d}(\bar\rho_S + \bar\rho_T)/\mathrm{d}z$ can be arbitrarily large, but smaller in absolute value than the numerator $\mathrm{d}\bar\rho_S/\mathrm{d}z$ in the case considered.

(3) *The contribution of the admixture to stratification is unstable, but the system is stabilized with a stable temperature stratification*: $\alpha\gamma_T > \beta\gamma_S > 0$, $0 < \eta < 1$. In this very interesting case, $m < 0$, but $|m| \to \infty$ as $\eta \to 1 - 0$. Hence, the final and initial perturbations have opposite signs; for instance, in response to initial heating, a cold trace is formed in the fluid, which can by far surpass the initial warming. This effect of 'negative heat capacity' in stratified two-component media [16, 17, 26, 27], at first glance curious, also has a simple explanation: for $\gamma_T > 0$ and the density stratification that is close to neutral, the developing intense vertical motions bring colder volumes of fluid from below.

The process of thermohaline trace formation can be clearly illustrated by considering the evolution of the initial temperature (buoyancy) distribution in the form of a modulated wave packet in the absence of salinity perturbations: for $t = 0$,

$$S' = 0\,, \quad T' = \varepsilon_T A(x)\cos kx \sin\lambda_n z\,, \quad \sigma = \alpha T\,, \quad \lambda_n = \frac{\pi n}{H}\,,$$

where $A(x)$ is the envelope slowly varying on the scale of the wavelength $2\pi/k$, ε_T is a small amplitude parameter, H is the fluid layer thickness, and n is the index of the vertical perturbation mode, $n = 1, 2, \ldots$. With such initial conditions, the invariant is $r = \gamma_S T_{\rm i}$. The asymptotic solution of Eqn (8) for the buoyancy can be written in the form

$$\begin{aligned}\sigma = 0.5\alpha\varepsilon_T \sin\lambda_n z\big[&A(x - c_{\rm g}t)\cos k(x - ct) \\ &+ A(x + c_{\rm g}t)\cos k(x + ct)\big]\,,\end{aligned}$$

where

$$c = \frac{N}{(k^2 + \lambda_n^2)^{1/2}}\,, \qquad c_{\rm g} = \frac{N\lambda_n^2}{(k^2 + \lambda_n^2)^{3/2}}\,.$$

are the respective phase and group velocities of the nth vertical mode of internal gravity waves.

This solution describes the breakup of the initial wave packet into two packets propagating in opposite directions from the region of the initial excitation (Fig. 1). As the packets leave this region, the buoyancy perturbation disappears within it, i.e., hydrostatic equilibrium sets in. In computations whose results are shown in Fig. 1, the shape of the envelope was specified as $A(x) = \exp[-(x/\Delta)^2]$ with $\Delta \gg k^{-1}$. The computations were performed for the value

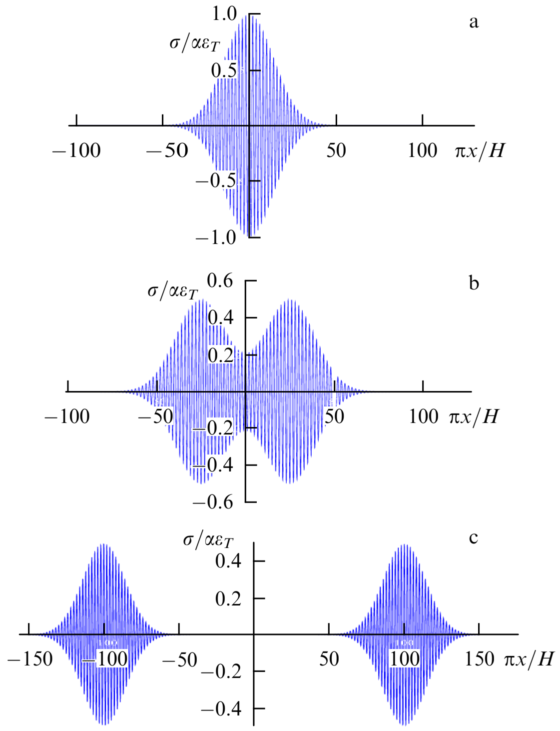

Figure 1. Horizontal distributions of buoyancy at the midlevel at successive time instants $t = 0$ (a), $t = 20$ (b), and $t = 40$ (c). The time scale is $H/\pi c$.

$k = 1/H$, which corresponds to the phase speed $c = NH/(\pi\sqrt{2})$ and the group speed $c_{\mathrm{g}} = 0.5c$. For $H = 5$ km and $N = 2\times10^{-3}$ s^{-1}, the phase speed is $c \approx 2$ m s^{-1}.

For known distributions of σ and r, the temperature and salinity perturbations are found from relations (12):

$$T' = \frac{\varepsilon_T}{1-\eta}\sin(\lambda_n z)\times \times\left[0.5\sum A(x\pm c_{\mathrm{g}}t)\cos k(x\pm ct) - \eta\,A(x)\cos kx\right], \tag{15}$$

$$S' = \frac{\alpha}{\beta}\frac{\varepsilon_T\eta}{1-\eta}\sin(\lambda_n z)\times \times\left[0.5\sum A(x\pm c_{\mathrm{g}}t)\cos k(x\pm ct) - A(x)\cos kx\right],$$

where $\sum$ denotes the sum of two components with the plus and minus signs. The plots of horizontal distributions (15) at the level $z = H/2$ at different time instants, shown in Figs 2 and 3 (for the first vertical mode), indicate that a stationary packet with zero density perturbation — the thermohaline trace — forms in the fields of temperature and salinity in the central part (the region of the initial perturbation), in addition to the propagating wave packets. We stress that the amplitude of this packet (the temperature perturbation amplitude) can substantially exceed the initial one; for the value $\eta = 2$ adopted in this example, the excess is exactly two-fold. Thus, an initial temperature perturbation can amplify (generally speaking, unlimitedly, depending on the value of the parameter η) in a medium with arbitrarily strong static stability!

We mention one more special feature of final perturbations. It follows from expressions (11) that discontinuities (jumps) in the initial distributions of T_{i} and S_{i} also persist at

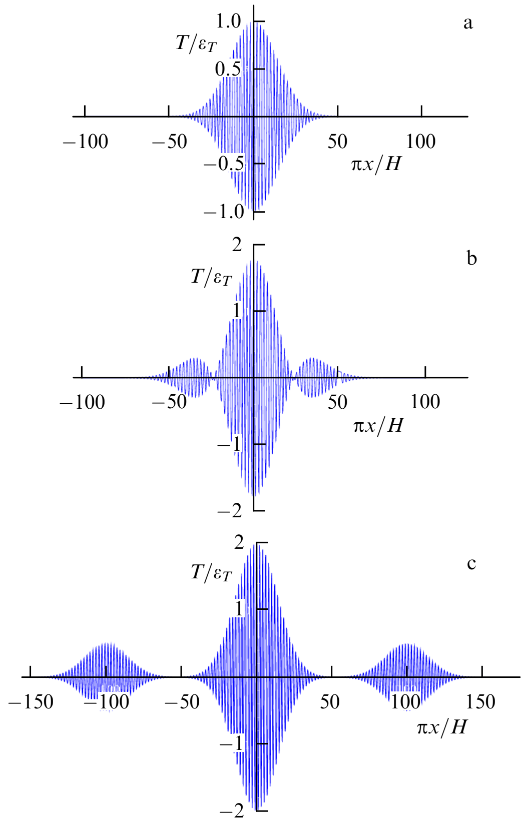

Figure 2. The same as in Fig. 1, but for the dimensionless temperature perturbation T/ε_T.

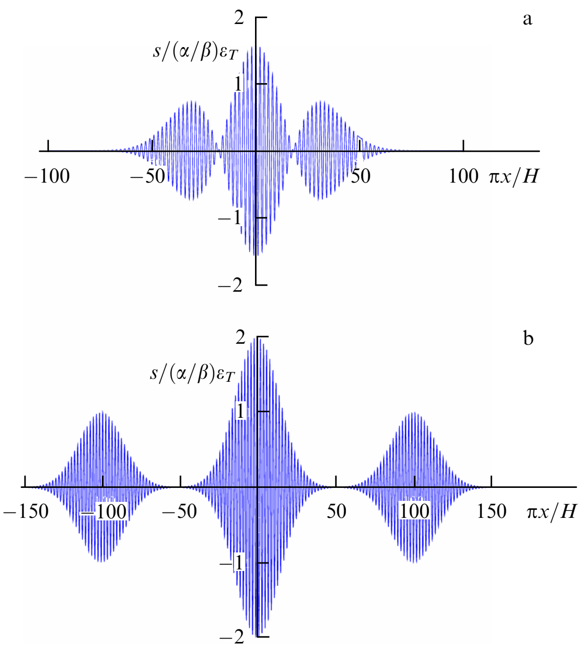

Figure 3. Horizontal distributions of the dimensionless salinity perturbation $S/(\alpha/\beta)\varepsilon_T$ at time instants $t = 20$ (a) and $t = 40$ (b). The salinity perturbation is absent at the initial instant $t = 0$.

the final stage. If a discontinuity initially exists in the field of one of the components (for instance, temperature), then, according to Eqn (11), it appears in the field of the other one (admixture concentration) in the process of hydrostatic adjustment. As is shown in Section 3.4, a universal feature of nonlinear dynamics consist in the formation of discontinuities from smooth initial conditions.

3.4 Nonlinear theory

We now explore the structure of final perturbations in a nonlinear problem. For simplicity, we consider two-dimensional motions occurring in the plane (x, z) and assume that $S_{\rm i} = 0$ at the initial instant and that the perturbation of temperature is localized horizontally, $T_{\rm i} \to 0$ as $|x| \to \infty$. The respective initial density distribution is written as $\rho_0(x, z) = \bar{\rho}(z) + \rho_{\rm i}(x, z)$, where $\bar{\rho} = \rho_*(1 - \gamma z)$ and $\rho_{\rm i} = -\alpha \rho_* T_{\rm i}$.

In the nonlinear case, the variables ρ, T, and S are Lagrangian invariants, i.e., the quantities preserved in fluid particles. Using this fact, we can uniquely determine the distributions of these variables in the final state. We first show that if the fluid reaches the state of mechanical equilibrium in the process of hydrostatic adjustment, then $\rho_{\rm f}(z) = \bar{\rho}(z)$ in this state, i.e., the final distribution coincides with the background one (the density perturbation disappears). Indeed, it follows from the definition of Lagrangian invariants that for any $t > 0$, $\rho = \rho_0(x_0, z_0)$, where x_0 and z_0 are the initial (Lagrangian) coordinates of a fluid particle. Hence, $\rho_{\rm f}(z) = \rho_0(x_0, z_0)$, i.e.,

$$\rho_{\rm f}(z) = \bar{\rho}(z_0) + \rho_{\rm i}(x_0, z_0) \,. \tag{16}$$

Passing in Eqn (16) to the limit $|x_0| \to \infty$ and taking into account that $\rho_{\rm i} \to 0$ and $z_0 \to z$ in that case, we obtain $\rho_{\rm f}(z) = \bar{\rho}(z)$.

We substitute $\rho_{\rm f}(z) = \bar{\rho}(z)$ in Eqn (16). Then it can be written in the form $\rho_*(1 - \gamma z) = \rho_*(1 - \gamma z_0) - \alpha \rho_* T_{\rm i}(x_0, z_0)$, which gives

$$z = z_0 + \frac{\alpha}{\gamma}\, T_{\rm i}(x_0, z_0) \,. \tag{17}$$

The physical meaning of formula (17) is quite transparent: it determines the dependence of the final (Eulerian) vertical coordinates of a fluid particle on its initial (Lagrangian) coordinates. If this dependence is known, the analogous dependence $x = x(x_0, z_0)$ for the horizontal coordinate can be found from the continuity equation in Lagrangian variables,

$$\frac{\partial(x, z)}{\partial(x_0, z_0)} = 1 \,. \tag{18}$$

Seeking a solution of Eqn (18) in the form $x = x(x_0, z_0)$, we obtain $\partial x / \partial x_0 = (\partial z / \partial z_0)^{-1}$, whence x is found by simple integration. Equations (17) and (18) therefore form a closed system for the field of Lagrangian displacements of fluid particles.

We now find the final temperature and admixture concentration distributions. From the definition of Lagrangian invariants and initial conditions (4), we find the expressions

$$T_{\rm f} = \gamma_T z_0 + T_{\rm i}(x_0, z_0) \,, \qquad S_{\rm f} = \gamma_S z_0 \,, \tag{19}$$

which, together with the dependences $x = x(x_0, z_0)$ and $z = z(x_0, z_0)$, yield a parametric representation (with x_0 and z_0 being the parameters) for the functions $T_{\rm f} = T_{\rm f}(x, z)$ and $S_{\rm f} = S_{\rm f}(x, z)$. Expressions (19) can be further transformed, taking into account that $z_0 = z - (\alpha/\gamma) T_{\rm i}$, in agreement with Eqn (17). Substituting this in Eqns (19) and recalling that $\alpha \gamma_T / \gamma = 1/(1 - \eta)$, we obtain

$$T_{\rm f} = \gamma_T z + \frac{\eta}{\eta - 1}\, T_{\rm i}(x_0, z_0) \,, \qquad S_{\rm f} = \gamma_S z + \frac{\alpha}{\beta}\, \frac{\eta}{\eta - 1}\, T_{\rm i}(x_0, z_0) \,. \tag{20}$$

We compare formulas (20) with the results of the linear theory. It can be readily seen that formulas (14) follow from Eqns (20) for deviations from the background, with the only difference that the left-hand and right-hand sides are now respectively expressed in Eulerian and Lagrangian coordinates. In the linear theory framework, $x_0 \sim x$ and $z_0 \sim z$, and formulas (14) and (20) are equivalent. In the nonlinear case, the dependence between the coordinates determined from Eqns (17) and (18) results in a rather intricate deformation of the initial distributions. Their most salient feature is the formation of discontinuities (jumps) in the vertical direction.

We next consider initial distributions of the form $T_{\rm i} = \Delta T h(x/L)\, \tau(z/H)$, where ΔT, L, and H are the amplitude and the horizontal and vertical scales of $T_{\rm i}$. The function $h(x)$, decaying at infinity, is chosen to be an even nonnegative function satisfying the condition $h(0) = 1$. According to Eqns (17) and (18), the dependences between the dimensionless Eulerian and Lagrangian coordinates (respectively normalized by L or H) become

$$z = z_0 + a h(x_0)\, \tau(z_0) \,, \qquad x = x_0 - a \int_0^{x_0} \frac{h(x_0)\, \tau'(z_0)}{1 + a h(x_0)\, \tau'(z_0)}\, {\rm d}x_0 \,, \tag{21}$$

where the dimensionless amplitude parameter $a = \alpha \Delta T / (\gamma H)$ is introduced and the prime denotes the derivative. We note that the expression for x in Eqns (21) is obtained by integrating the equation $\partial x / \partial x_0 = (\partial z / \partial z_0)^{-1}$ using the symmetry condition: $x = 0$ for $x_0 = 0$. It is assumed that $z_0 = z_0(x_0, z)$ (the expression for z is solved for z_0) in the integrand in Eqns (21).

We begin with a simple example in which the relations between coordinates can be expressed explicitly. Let the fluid be confined to the layer $0 < z < 1$ (the coordinates are dimensionless hereinafter) and $\tau(z_0)$ is the piecewise linear function

$$\tau(z_0) = \begin{cases} z_0 \,, & 0 < z_0 < 0.5 \,, \\ 1 - z_0 \,, & 0.5 < z_0 < 1 \end{cases}$$

(the temperature perturbation attains a maximum in the layer center and is absent at the boundaries). From relations (21), we find

$$\begin{aligned} & z = z_0 \big(1 + a h(x_0)\big) \,, \\ & x = x_0 - a \int_0^{x_0} \frac{h(x_0)}{1 + a h(x_0)}\, {\rm d}x_0 \,, \qquad 0 < z_0 < 0.5 \,, \\ & z = z_0 + a h(x_0)(1 - z_0) \,, \\ & x = x_0 + a \int_0^{x_0} \frac{h(x_0)}{1 - a h(x_0)}\, {\rm d}x_0 \,, \qquad 0.5 < z_0 < 1 \,. \end{aligned} \tag{22}$$

For $a > 0$, formulas (22) describe fluid particle displacements as the fluid convectively ascends and then diverges in the vicinity of the upper boundary. Expressing the coordinate

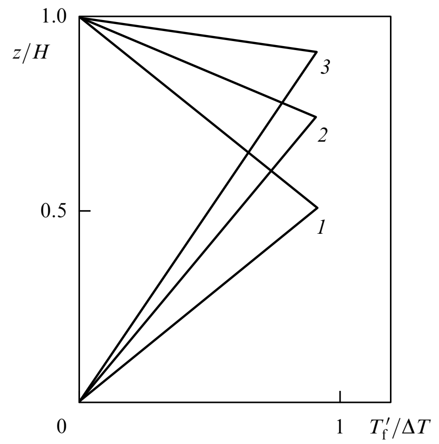

Figure 4. Deviations of temperature from the linear background distribution for various values of the parameter a and a piecewise linear dependence $\tau(z_0)$ with $\eta = 2$: $a \ll 1$ (*1*), $a = 0.5$ (*2*), and $a = 0.8$ (*3*).

z_0 in terms of z from Eqns (22) and substituting it in Eqn (20), we find the deviation from the background linear distribution $T_f' = T_f - \gamma_T z$ in the form

$$T_f' = \Delta T \frac{\eta}{\eta - 1} h(x_0) \begin{cases} \dfrac{z}{1+a}, & 0 < z < 0.5(1+a), \\ \dfrac{1-z}{1-a}, & 0.5(1+a) < z < 1. \end{cases} \quad (23)$$

Dependence (23) is presented in Fig. 4 for $x_0 = x = 0$ and several values of the parameter a. This figure highlights the tendency to the formation of discontinuities in the final vertical distributions: as a increases, the initial triangular distribution tends to 'break' in the upper layer half.

We now explore the structure of vertical temperature distributions for arbitrary smooth dependences $\tau(z_0)$. According to Eqns (19) and (21), the distribution $T_f = T_f(z)$ at the symmetry axis $x_0 = 0$ of the temperature perturbation is described by the parametric expressions (z_0 is the dimensionless parameter)

$$z = z_0 + a\tau(z_0), \qquad T_f = \gamma_T H\big[z_0 + a(1-\eta)\,\tau(z_0)\big], \quad (24)$$

where we took into account that $\Delta T/(\gamma_T H) = a(1-\eta)$. Figure 5a plots dependence (24) for various values of a in the case of a smooth localized perturbation $\tau(z_0) > 0$ and $\eta < 0$ (a vertically unbounded medium is considered). As can be seen from the figure, as a increases, a discontinuity forms in the vertical profile T_f at some $a = a_{cr}$ at a point $z = z_{cr}$. Similarly, a discontinuity also forms in the vertical profile of the admixture concentration (Fig. 5b). These discontinuities compensate each other such that the vertical density distribution remains continuous (linear).

We determine the critical parameter values following Ref. [28]. In the same fashion as in Ref. [28], it can be shown that the formation of a discontinuity is associated with the appearance of an inflection point z_* on the plot of the dependence $z = z(z_0)$. This point is found from the equation $\tau''(z_0) = 0$ (here, primes denote derivatives). Because $\partial T/\partial z = (\partial T/\partial z_0)/(1 + a\tau'(z_0))$, we find the critical parameters

$$a_{cr} = -\frac{1}{\tau'(z_*)}, \qquad z_{cr} = z_* + a_{cr}\tau(z_*).$$

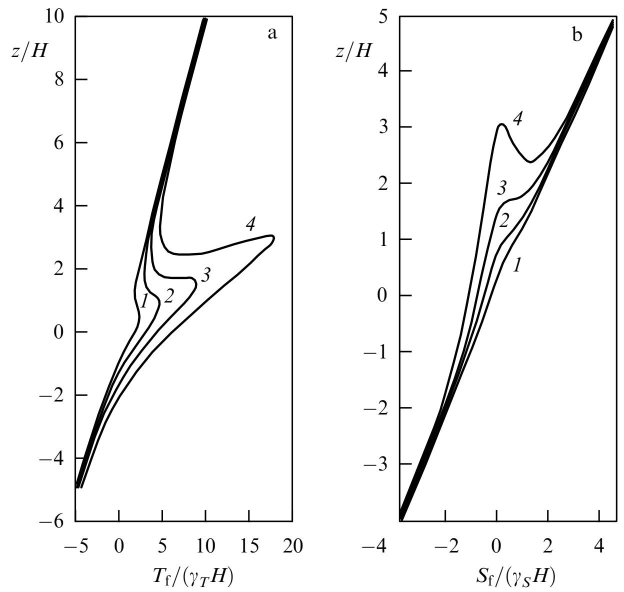

Figure 5. (a) Vertical distribution of temperature and (b) admixture concentration at the symmetry axis of the perturbation for $\tau(z_0) = 1/(1+z_0^2)$, $\eta = -5$, and various values of the parameter a: $a = 0.4$ (*1*), $a = 0.8$ (*2*), $a = a_{cr} = 8\sqrt{3}/9 \approx 1.54$ (*3*), $a = 3$ (*4*).

In particular, for $\tau(z_0) = 1/(1+z_0^2)$, elementary manipulations give $a_{cr} = 8\sqrt{3}/9 \approx 1.54$, $z_{cr} = \sqrt{3}$. For the profile $\tau(z_0) = \exp(-z_0^2)$, we obtain $a_{cr} = \exp(0.5)/\sqrt{2} \approx 1.17$, $z_{cr} = \sqrt{2}$. The vertical temperature and admixture concentration distributions are continuous for $a < a_{cr}$ and multi-valued (discontinuous) for $a > a_{cr}$. Using the above values, we can estimate the critical amplitude ΔT above which a discontinuity occurs: $\Delta T = a_{cr}\gamma H/\alpha$ (this quantity linearly depends on the perturbation scale H). For the profile $\tau(z_0) = 1/(1+z_0^2)$ and the values $H = 10$ m, $\alpha = 2\times10^{-4}$ K^{-1}, and $\gamma = 9\times10^{-7}$ m^{-1} ($N = 3\times10^{-3}$ s^{-1}) characteristic for the ocean, we obtain $\Delta T = 7\times10^{-2}$ K, which shows that the initial perturbation of a fraction of a degree in amplitude can already lead to the formation of a discontinuity.

We emphasize that the effect of discontinuity formation bears a universal character and is not related to the details of $\tau(z_0)$. Its physical explanation is quite simple. Indeed, according to Eqn (17), the vertical displacement $l = z - z_0$ of a fluid particle is determined by the magnitude ΔT of the initial perturbation T_i. For $\Delta T > 0$, warmer particles, rising, overtake the colder ones, which results in the formation of a discontinuous profile resembling the known N-wave (the simple Riemann wave) of gasdynamics [29]. Such profiles are quite frequently detected in ocean measurements; they are characteristic of so-called temperature inversions [22, 23].

Dependence (24) for $\tau(z_0) = \cos(\lambda z_0)$ and two values of the parameter a are plotted in Fig. 6, from which it can be seen that in the case where the initial perturbation is periodic in the vertical direction (for example, due to an internal wave), a stationary sawtooth temperature profile can form, in which layers with uniform gradients are separated by sharp jumps (inversions). The condition for the gradients to form is $a > a_{cr} = 1/\lambda$ for a periodic perturbation; the related critical magnitude of ΔT for $\lambda = 1$ and the parameters listed above is 4.5×10^{-2} K. If the initial distribution is aperiodic (has phases of both signs), then, obviously, a profile with an irregular distribution of discontinuities emerges. Such profiles are characteristic of the ocean fine structure, which,

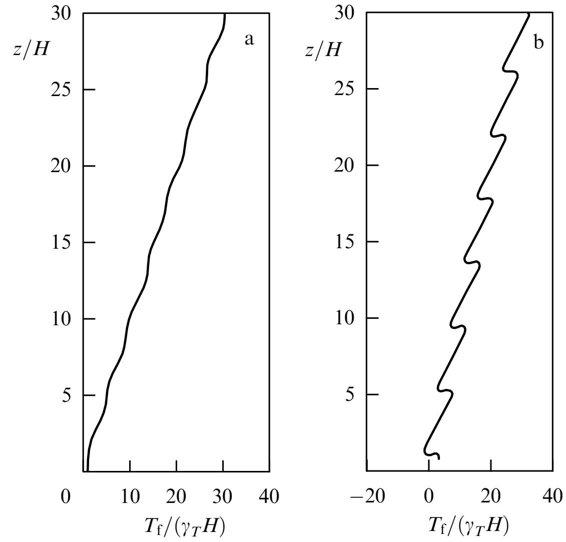

Figure 6. Vertical distribution of temperature at the symmetry axis for $\tau(z_0) = \cos(\lambda z_0)$, $\lambda = 1.5$, and $\eta = -2$ and the parameter values $a = 0.2$ (a) and $a = 0.95 > a_{cr}$ (b).

according to the definition proposed in Ref. [22, p. 119], "represents disordered or systematic alteration with depth of the regions with low and high vertical gradients of one or another property." We note that for the fine structure, the characteristic values of jumps in temperature attain several tenths of a degree, which corresponds to supercritical regimes.

3.5 Integral and Lagrangian conservation laws for the system of equations of adiabatic dynamics

The hydrodynamic memory of stratified two-component media is tightly connected with the existence of a number of specific integral and Lagrangian conservation laws. We consider some of them in this section.

We first elaborate on the hydrodynamical meaning of local conservation law (11) responsible for the formation of a trace in the linear theory. As noted above, in the nonlinear case, the variables T and S are Lagrangian invariants — quantities that are conserved in fluid particles. Hence, it immediately follows that the intersection lines of surfaces $T = \text{const}$ and $S = \text{const}$ always consist of the same particles, i.e., they are the fluid lines. We introduce the vector field $\mathbf{R}(\mathbf{x}, t)$ tangent to the intersection lines of these surfaces:

$$\mathbf{R}(\mathbf{x}, t) = \nabla T \times \nabla S \,. \tag{25}$$

It follows from Eqns (3) that the field $\mathbf{R}(\mathbf{x}, t)$ satisfies the equation $\partial \mathbf{R}/\partial t = \operatorname{rot}(\mathbf{u} \times \mathbf{R})$, which, after using a known vector identity, can be rewritten as

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = (\mathbf{R}\nabla)\mathbf{u} \,. \tag{26}$$

In hydrodynamics, Eqn (26) is known to describe the evolution of a vector field frozen in the fluid [30, 31]. If Eqn (26) is satisfied, then the vector lines of the field $\mathbf{R}$ move with fluid particles and the intensity of the vector tubes is preserved.

We demonstrate that conservation law (9) represents a linearized form of Eqn (26) of the frozen-in vector field evolution. Indeed, substituting $T = \bar{T} + T'$ and $S = \bar{S} + S'$ in Eqn (25), for small deviations, we find $\mathbf{R} = (\gamma_T \mathbf{k} + \nabla T') \times (\gamma_S \mathbf{k} + \nabla S') \sim \mathbf{R}' = \nabla r \times \mathbf{k}$. In the linear approximation, Eqn (26) is written as $\partial \mathbf{R}/\partial t = 0$. For perturbations bounded at infinity, the last (vector) equation is equivalent to Eqn (9).

We note that in addition to T and S, one more Lagrangian invariant of the system of equations of adiabatic dynamics (1)–(3) is Ertel's potential vorticity [30],

$$\frac{\mathrm{d}q}{\mathrm{d}t} = 0 \,, \qquad q = \operatorname{rot}\mathbf{u}\nabla\rho \,.$$

Equation (26) for the frozen-in vector field evolution is obviously satisfied by the field $\mathbf{R}$ composed of an arbitrary pair of invariants from the set T, S, and q, for example, $\mathbf{R} = \nabla\rho \times \nabla q$. It also implies interesting integral conservation laws. It can be easily verified that for the vector fields $\mathbf{u}$ and $\mathbf{R}$ respectively satisfying system (1)–(3) and Eqn (26), the following formula is valid:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}\mathbf{u}) = \operatorname{div}(L\mathbf{R}) + p\mathbf{R}\nabla\left(\frac{1}{\rho}\right), \qquad L = \frac{\mathbf{u}^2}{2} - \frac{p}{\rho} - gz \,. \tag{27}$$

We set $\mathbf{R} = \nabla\rho \times \nabla q$ in Eqn (27). Then the second term in the right-hand side vanishes and the following law emerges after integration over the volume V:

$$\frac{\partial I_1}{\partial t} = 0 \,, \qquad I_1 = \int_V (\nabla\rho \times \nabla q)\mathbf{u}\,\mathrm{d}\mathbf{x} \,. \tag{28}$$

Here, it is assumed that the boundary of the integration domain is composed of segments tangent to $\mathbf{R}$.

Similarly, taking $\mathbf{R} = \nabla T \times \nabla S$ in Eqn (27) and integrating, we obtain another law:

$$\frac{\partial I_2}{\partial t} = 0 \,, \qquad I_2 = \int_V [\nabla T \times \nabla S]\mathbf{u}\,\mathrm{d}\mathbf{x} \,. \tag{29}$$

An arbitrary function $\mu(T, S, q)$ can be included in the integrands in Eqns (28) and (29) [the field $\mu\mathbf{R}$ satisfies Eqn (26) if $\mathbf{R}$ satisfies it].

In the general case of motions in a compressible one-component fluid, an analog of conservation law (28) was obtained in Ref. [32]. Conservation law (29), specific for two-component media, was derived in Ref. [33] in the framework of a Hamiltonian description of dynamics (the Casimir invariant C_3). Conservation laws (28) and (29), analogous to the conservation laws of Moffat and Volterra in hydrodynamics and magnetohydrodynamics [30, 32], are independent of the energy conservation law

$$\frac{\partial E}{\partial t} = 0 \,, \qquad E = \int_V \left[\frac{\mathbf{u}^2}{2} + gz\right]\rho\,\mathrm{d}\mathbf{x} \,.$$

We mention one more important feature of the adiabatic dynamics of two-component media, related to the Cauchy problem for system (1)–(3). When solving this problem, it is convenient to reduce Eqns (3) to a single equation for the density. The fields $\mathbf{u}$, p, and ρ can then be found by solving the 'shortened' system

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p - g\mathbf{k} \,, \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \,, \qquad \operatorname{div}\mathbf{u} = 0 \,, \tag{30}$$

which coincides with the standard system of equations describing the dynamics of an incompressible one-component medium. When these fields are found, a question arises as to how to recover the temperature and admixture concentration fields. As is demonstrated in Ref. [16], the fields T and S can be uniquely reconstructed at an arbitrary time instant from the known density and velocity fields and the initial data. A description of the dynamics of two-component media can therefore be carried out on the basis of Eqns (30), reconstructing the temperature and concentration fields at each time step.

In summary, we have shown that long-lived 'traces' can form in the temperature and admixture concentration fields in binary mixtures, including the formation of discontinuous distributions of these fields. Additionally, we demonstrated that thermal perturbations can intensify in hydrostatically stable media and can also change their sign (for example, showing reduced temperatures in response to supplied heat or, equivalently, exhibiting an effective 'negative heat capacity').

We note that temperature and salinity inhomogeneities, compensating each other in the density field, are systematically observed in the ocean (see, e.g., Refs [34–36]). In the oceanographic literature, they are called 'spices'. The mechanism of hydrostatic adjustment described above explains the origin and widespread occurrence of spices in a natural way.

4. Formation of compensated inhomogeneities in shear flows

Alongside the mechanism of hydrostatic adjustment considered in Section 3, one more mechanism of the formation of compensated (T, S) distributions (compensated traces) can be related to the specific features of perturbation dynamics in hydrodynamically stable shear flows [37]. In such flows, perturbations in the fields of buoyancy (density) and velocity decay with time, whereas the compensated perturbations in temperature and admixture concentration (salinity) are carried by flows of the ideal fluid without decay, forming moving compensated inhomogeneities.

For simplicity, we restrict ourself to two-dimensional perturbations in a plane-parallel flow with a vertical shear, $\mathbf{u} = U(z)\mathbf{i}$, where $\mathbf{i}$ is a horizontal unit vector. The behavior of small-amplitude perturbations (the primes are omitted as long as this does not lead to a confusion) is governed by the system of equations [30, 38]

$$\frac{\partial u}{\partial t} + U(z)\,\frac{\partial u}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}z}\,w = -\frac{1}{\rho_*}\,\frac{\partial p}{\partial x}\,, \qquad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\,, \tag{31}$$

$$\frac{\partial w}{\partial t} + U(z)\,\frac{\partial w}{\partial x} = -\frac{1}{\rho_*}\,\frac{\partial p}{\partial z} + g(\alpha T - \beta S)\,,$$

$$\frac{\partial T}{\partial t} + U(z)\,\frac{\partial T}{\partial x} + \gamma_T w = 0\,, \quad \frac{\partial S}{\partial t} + U(z)\,\frac{\partial S}{\partial x} + \gamma_S w = 0\,, \tag{32}$$

which is supplemented by initial conditions (4).

The general decomposition (12) of thermohaline perturbation fields into sums of density-related and compensated components is also valid for solutions of system (31), (32). According to Eqn (12), these components can be uniquely determined from the invariant $r = \gamma_S T - \gamma_T S$ and the buoyancy $\sigma = -\rho'/\rho_* = \alpha T - \beta S$. In a shear flow, the evolution of r obeys the simple equation $\partial r/\partial t + U(z)\,\partial r/\partial x = 0$, which, as above, follows by eliminating w from Eqns (32). The solution with the initial condition $r|_{t=0} = r_{\mathrm{i}}(x,z) = \gamma_S T_{\mathrm{i}} - \gamma_T S_{\mathrm{i}}$ takes the form $r = r_{\mathrm{i}}(x - U(z)t, z)$. Accordingly, in a shear flow, the compensated component $T_r = -\gamma^{-1}\beta r$, $S_r = -\gamma^{-1}\alpha r$ plays the role of a passive tracer carried by the flow. This component propagates without decay, experiencing only shear deformations.

The reduction of Eqns (32) to a single buoyancy equation leads to a closed (shortened) system for the buoyancy and velocity fields. In terms of the streamfunction $u = -\partial\psi/\partial z$, $w = \partial\psi/\partial x$, the system becomes

$$\frac{D}{Dt}\,\Delta_2\psi - U''(z)\,\frac{\partial\psi}{\partial x} = g\,\frac{\partial\sigma}{\partial x}\,, \qquad \frac{D}{Dt}\,\sigma + \frac{N^2}{g}\,\frac{\partial\psi}{\partial x} = 0\,, \tag{33}$$

where Δ_2 is the two-dimensional Laplace operator and $D/Dt = \partial/\partial t + U\partial/\partial x$. This system coincides with that for perturbative dynamics in a one-component medium and is the main object of the theory of hydrodynamic stability of shear flows in a stratified fluid [30, 38]. The classical result of this theory, obtained on the basis of the normal mode method, states that the instability is possible only for the Richardson numbers $\mathrm{Ri} = N^2/U'^2 < 1/4$. For $\mathrm{Ri} > 1/4$ (the stability condition), the general solution of system (33) is a superposition of neutral modes of the discrete spectrum (propagating without decay) and decaying modes of the continuous spectrum. For a broad class of smooth shear flows, discrete spectrum modes are absent. Perturbations of the buoyancy and velocity fields in this case decay with time and, accordingly, the density-related component T_σ, S_σ of the thermohaline perturbation field also decay. By contrast, the compensated component is carried by the flow without decay, forming a moving compensated inhomogeneity. This is the essence of the mechanism ensuring the formation of compensated inhomogeneities in hydrodynamically stable shear flows.

As an illustration of the outlined mechanisms, we consider the dynamics of perturbations in an unbounded flow with a uniform vertical shear $U(z) = dz$, $d = \mathrm{const}$. We assume that a harmonic temperature perturbation is specified at the initial instant $(t = 0)$, while perturbations of salinity and velocity are absent:

$$\psi_{\mathrm{i}} = 0\,, \quad S_{\mathrm{i}} = 0\,, \quad T_{\mathrm{i}} = \varepsilon_T \cos(kx + mz) \quad \text{at} \quad t = 0\,.$$

For these initial conditions, the compensated component in the field of thermohaline perturbations is given by

$$T_r = \frac{\eta}{\eta - 1}\cos\theta\,, \qquad S_r = \frac{\alpha}{\beta}\,\frac{\eta}{\eta - 1}\cos\theta\,, \qquad \theta = kx + (m - dkt)z\,. \tag{34}$$

The perturbations of buoyancy and velocity fields are found from system (33) with the initial conditions $\psi_{\mathrm{i}} = 0$ and $\sigma_{\mathrm{i}} = \alpha T_{\mathrm{i}}$. The solution is written as $\sigma = \hat\sigma(t)\cos\theta$, $\psi = \hat\psi(t)\sin\theta$, where the amplitude functions $\hat\sigma(t)$ and $\hat\psi(t)$ satisfy the system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[k^2 + (m - dkt)^2\right]\hat\psi - gk\hat\sigma = 0\,, \qquad \frac{\mathrm{d}\hat\sigma}{\mathrm{d}t} + \frac{N^2}{g}\,k\hat\psi = 0\,, \tag{35}$$

with the conditions $\hat\psi(0) = 0$ and $\hat\sigma(0) = \alpha\varepsilon_T$. Passing to the dimensionless time $\tau = dt$, we reduce system (35) to the single equation

$$\frac{\mathrm{d}^2\hat\xi}{\mathrm{d}\tau^2} + \mathrm{Ri}\,\omega^2(\tau)\hat\xi = 0\,, \qquad \omega^2(\tau) = \left[1 + (\delta - \tau)^2\right]^{-1},$$

where $\hat{\xi} = [1 + (\delta - \tau)^2]\hat{\psi}$ is the dimensionless perturbation of relative vorticity, $\delta = m/k$, and $\mathrm{Ri} = N^2/d^2$ is the Richardson number. For $\mathrm{Ri} \gg 1$, the standard Wentzel–Kramers–Brullouin (WKB) method leads to the asymptotic solution [38]

$$\hat{\xi}(t) = \alpha\varepsilon_T \frac{\sqrt{\mathrm{Ri}}}{k\sqrt{\omega(\tau)}} \sin\left(\sqrt{\mathrm{Ri}} \int_0^\tau \omega(\tau)\,\mathrm{d}\tau\right), \quad \tau = dt,$$
$$\hat{\sigma}(t) = \alpha\varepsilon_T \sqrt{\frac{\omega(\tau)}{\omega(0)}} \cos\left(\sqrt{\mathrm{Ri}} \int_0^\tau \omega(\tau)\,\mathrm{d}\tau\right),$$

valid up to the terms $O(1/\sqrt{\mathrm{Ri}})$. From these expressions it directly follows that in shear flows, perturbations of the buoyancy and velocity fields decay as $1/\sqrt{t}$. The density perturbation follows this same law. Hence, at large times, the resulting solution preserves only the perturbations of temperature and salinity (34) compensating each other in the density field.

5. Anomalous responses to mechanical and thermal surface forcing

5.1 The problem of inhomogeneous tangent stresses and preliminary estimates

This section deals with the response of a stratified binary mixture to inhomogeneous tangent stresses on its surface [39]. This problem is well studied for one-component fluids, being of substantial interest for a number of applications. They include, first, certain well-known geophysical problems pertaining to the reaction of the upper layer of water bodies to inhomogeneous wind stresses [40]. Another example is related to certain mechanisms of convective instability in two-layer systems, discussed recently ('anticonvection' [41]). In such systems, convective motions in one of the contacting media generate horizontally inhomogeneous tangent stresses at the interface, which excite flows and thermal perturbations in the vertically adjacent layer of the stratified medium, which can support convection in the first medium, thus establishing a positive feedback. One of the most important elements of dynamics in such systems is the response of the stratified medium to inhomogeneous tangent stresses at its horizontal boundary. To an even greater extent, this pertains to questions of thermocapillary convection [42, 43]. Paper [39] draws attention to nontrivial aspects of the reaction of two-component media to mechanical actions of that kind.

The problem geometry is schematically depicted in Fig. 7. We consider a semi-infinite fluid layer $z \leqslant 0$ (the z axis is directed vertically upward), stratified in temperature and admixture concentration (salinity).

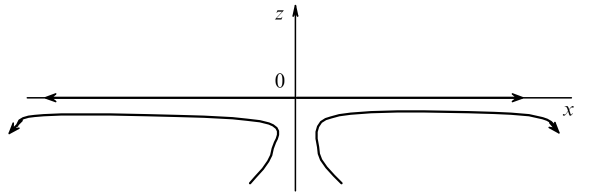

Figure 7. The geometry of the problem. The bold arrows schematically show inhomogeneous tangent stresses at the horizontal surface of the fluid and streamlines of the associated flows.

The linearized stationary system of equations for perturbations in the Boussinesq approximation is [1–3]

$$0 = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\mathbf{v} + g(\alpha T - \beta S)\mathbf{e}_z, \qquad \nabla\mathbf{v} = 0,$$
$$\gamma_T \mathbf{v}\mathbf{e}_z = \kappa\nabla^2 T, \qquad \gamma_S\mathbf{v}\mathbf{e}_z = \chi\nabla^2 S, \tag{36}$$

where $\mathbf{v}$ is the vector of velocity field perturbations, $\mathbf{e}_z$ is a unit vertical vector along the z axis, ρ_0 is the mean (reference) medium density, ν is the kinematic viscosity, κ is the coefficient of thermal conductivity, and χ is the diffusivity of the admixture.

We suppose that stationary tangent stresses harmonically depending on the horizontal coordinate x are applied at the fluid surface at $z = 0$ [for simplicity, we restrict ourself to a two-dimensional problem in the plane (x, z)],

$$\rho_0\nu\frac{\partial u}{\partial z} = E\sin kx \quad \text{at} \quad z = 0, \tag{37}$$

where E is the amplitude of stresses, u is the horizontal velocity component, and k is the wavenumber. We neglect surface deformations, and therefore the vertical velocity component w vanishes at the surface $z = 0$. The temperature and admixture concentration on the surface are supposed to satisfy boundary conditions of the third kind:

$$\frac{\partial T}{\partial z} = -\frac{T}{h_T}, \qquad \frac{\partial S}{\partial z} = -\frac{S}{h_S} \qquad \text{at} \quad z = 0, \tag{38}$$

where nonnegative quantities h_T and h_S are fixed length scales. We assume that all perturbations decay far from the surface (at $z \to -\infty$).

Inhomogeneous tangent stresses at the surface excite horizontally inhomogeneous flows in the fluid. If the horizontal velocity component u depends on the horizontal coordinate x, vertical motions must be generated because of continuity constraints. In a stratified medium, they transport both heat and admixture, leading to perturbations in both. We let H denote the depth (yet unknown) of penetration of stationary perturbations into the stably stratified medium. Using boundary condition (37), we can estimate the velocity amplitude of the excited horizontal flows as

$$\frac{\rho_0\nu u}{H} \sim E, \qquad u \sim \frac{EH}{\rho_0\nu} \tag{39}$$

(for simplicity, the amplitude of velocity perturbations is also denoted by u). The continuity requires that

$$ku \sim \frac{w}{H}, \qquad w \sim Hku \sim \frac{EH^2k}{\rho_0\nu}. \tag{40}$$

The amplitude of the related temperature perturbation can be estimated from the heat conductance equation as

$$\gamma_T w \sim \frac{\kappa T}{H^2}, \qquad T \sim \frac{\gamma_T w H^2}{\kappa} \sim \frac{\gamma_T k E H^4}{\rho_0\kappa\nu}. \tag{41}$$

Here, only the 'vertical' part of the Laplacian in the heat conductance equation is taken into account. This is appropriate because, as can be easily seen from the analysis in what follows, the vertical scale H of the arising perturbations cannot exceed their horizontal scale $L = 2\pi/k$ by an order of magnitude. For the same reason, for rough estimates, the hydrostatic approximation can be invoked, yielding

$$p' \sim g\rho' H \sim g\rho_0 |\alpha T - \beta S| H\,, \qquad (42)$$

where the primes denote perturbation amplitudes for the pressure and density. Additionally, in the horizontal projection of the equations of motion, the leading terms can be equated by an order of magnitude: $\nu\,\partial^2 u/\partial z^2 \sim \nu u/H^2$ and $(1/\rho_0)(\partial p'/\partial x) \sim g|\alpha T - \beta S| kH$. This gives

$$|\alpha T - \beta S| \sim \frac{E}{\rho_0 g k H^2}\,. \qquad (43)$$

In the case of a one-component medium (neglecting S), Eqns (41) and (43) lead to the following estimates for the perturbation penetration depth H and amplitude T:

$$H \sim \left(\frac{\nu\kappa}{\alpha g \gamma_T k^2}\right)^{1/6}, \qquad T \sim \frac{E}{\rho_0 \alpha g k H^2} \sim \frac{E}{\rho_0}\left(\frac{\gamma_T}{\nu\kappa k(\alpha g)^2}\right)^{1/3}. \qquad (44)$$

The solution for the usual one-component medium, which is only temperature stratified, leads indeed to this result [40], derived here from simple physical considerations (scaling analysis). The vertical scale H expressed by relation (44) has been known since long ago (in the western literature, it is sometimes referred to as the Lineikin scale in recognition of the well-known Russian oceanographer [44]). This is the depth reached by linear stationary perturbations of various natures in a stably stratified medium [44, 40]. The scale H depends on the buoyancy frequency (the Brunt–Väisälä frequency) $N_T = (\alpha g \gamma_T)^{1/2}$. If we turn to a two-component medium (i.e., take the contribution of the admixture concentration to the density stratification into account), the buoyancy frequency also changes in general. At first glance, we only need to introduce required changes to this frequency (or, equivalently, to the density stratification). According to Eqn (44), the dependence on stratification is weak ($H \sim N_T^{-1/3}$), and it therefore seems that additionally taking the admixture stratification into account would lead to only small quantitative corrections. But the solution in Ref. [39] shows that in reality, accounting for a two-component character of the medium generally changes the results much more dramatically. The temperature response to the mechanical action can principally differ from the one predicted by the apparently obvious scaling analysis.

5.2 The solution and its analysis

The solution of the formulated linear problem does not involve any fundamental difficulties but is somewhat cumbersome. It is worth devoting some time to it because it leads to nontrivial results and, besides, a similar system is used to solve other informative problems considered in what follows.

We seek stationary solutions for perturbations that depend on the horizontal coordinate x harmonically:

$$u(x,z) = U(z)\sin(kx)\,, \qquad w(x,z) = W(z)\cos(kx)\,,$$
$$T(x,z) = \theta(z)\cos(kx)\,, \qquad S(x,z) = \vartheta(z)\cos(kx)\,,$$

and so on. Eliminating all unknowns except w from Eqns (36), it is straightforward to obtain the equation

$$\left(\frac{d^2}{dz^2} - k^2\right)^3 W = k^6 \mathrm{R} W\,, \qquad (45)$$

where

$$\mathrm{R} = \frac{1}{k^4\nu}\left(\frac{N_T^2}{\kappa} + \frac{N_S^2}{\chi}\right) = \frac{N_T^2}{k^4\kappa\nu}\left(1 + \frac{\kappa N_S^2}{\chi N_T^2}\right) = \mathrm{R}_T(1-\xi) = \mathrm{R}_T + \frac{\mathrm{R}_S}{\delta}\,, \qquad (46)$$

and $N_T^2 = \alpha g \gamma_T$ and $N_S^2 = -\beta g \gamma_S$ are the thermal and saline contributions to the squared buoyancy frequency $N^2 = N_T^2 + N_S^2$; the dimensionless parameters $\mathrm{R}_T = N_T^2/(\nu\kappa k^4)$ and $\mathrm{R}_S = N_S^2/(\nu\kappa k^4)$ introduced here are analogs of the Rayleigh numbers, and

$$\xi \equiv \frac{\kappa}{\chi}\frac{\beta\gamma_S}{\alpha\gamma_T} = -\frac{\kappa}{\chi}\frac{N_S^2}{N_T^2} = -\frac{1}{\delta}\frac{N_S^2}{N_T^2}\,, \qquad \delta \equiv \frac{\chi}{\kappa}\,. \qquad (47)$$

The dimensionless parameter R has the meaning of a generalized Rayleigh number. Instead of the fluid layer thickness (which is infinite in the problem under study), it involves the horizontal perturbation scale k^{-1}. For the region of admissible values of γ_T and γ_S (the region of convective stability), R is nonnegative.

We seek a solution of Eqn (45) in the form of a linear combination of exponentials $\exp(q_i k z)$, where q_i are roots of the characteristic equation

$$(q^2 - 1)^3 = \mathrm{R}\,. \qquad (48)$$

From the six exponentials, only three decay as $z \to -\infty$, and the coefficients at the others should be set to zero. The solution for the vertical velocity can be written as

$$W(z) = \sum_{i=1}^{3} C_i \exp(q_i k z)\,, \qquad (49)$$

$$q_1 = (1 + \mathrm{R}^{1/3})^{1/2}\,, \qquad q_{2,3} = \left[1 + \mathrm{R}^{1/3}\exp\left(\pm\frac{2}{3}\pi \mathrm{i}\right)\right]^{1/2}. \qquad (50)$$

where C_i are integration constants and $\mathrm{Re}\, q_i > 0$. It what follows, we restrict ourself to the analysis of the solution in the case $\mathrm{R} \gg 1$ (more precisely, a stronger inequality $\mathrm{R}^{1/6} \gg 1$ is assumed to hold). This corresponds to situations where the stable density stratification is rather strong and governs the structure of perturbations arising in the fluid to a substantial degree. In such cases, unity can be neglected compared with other terms in Eqns (48) and (50), giving

$$q_1 \approx \mathrm{R}^{1/6}\,, \qquad q_{2,3} \approx \mathrm{R}^{1/6}\exp\left(\pm\frac{1}{3}\pi \mathrm{i}\right), \qquad (51)$$

and the solution simplifies considerably. The exponents in Eqn (49) (and also in the expressions for u and perturbations of pressure and buoyancy) contain the large factor $\mathrm{R}^{1/6}$, implying that the perturbations of these quantities decay rapidly with depth (at vertical distances that are much smaller than the horizontal perturbation scale $L = 2\pi/k$). In other words, the aspect ratio (the ratio of characteristic horizontal to vertical scales) for these perturbations is much greater than unity. For that reason, the derivatives with respect to the horizontal coordinates x are negligibly small compared to the z-derivatives in the velocity Laplacian in the first equation in (36). Because the perturbations are stretched horizontally, the characteristic vertical velocity is much smaller than the horizontal one. Accordingly, as follows from the scaling analysis, the velocity Laplacian can be neglected altogether in the vertical projection of the above equation (which means that we are limited to the hydrostatic approximation, which is applicable for perturbations stretched horizontally). Hence, in the approximation adopted here, the two projections of the dynamic equation have the form

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\,\frac{\partial^2 u}{\partial z^2}\,, \qquad 0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g(\alpha T - \beta S)$$

(in fact, these simplifications have already been used above in the preliminary estimates).

Comparing expressions for temperature and admixture concentration perturbations with Eqn (49) readily shows that they, generally speaking, also contain the fourth exponential, $\exp(kz)$. From heat and admixture transport equations (36), it is rather straightforward to obtain the expressions

$$T = \frac{\gamma_T}{\kappa k^2}\left[C_T \exp(kz) + \sum_{i=1}^{3}\frac{C_i}{q_i^2 - 1}\exp(kq_i z)\right]\cos(kx)\,,$$

$$S = \frac{\gamma_S}{\chi k^2}\left[C_S \exp(kz) + \sum_{i=1}^{3}\frac{C_i}{q_i^2 - 1}\exp(kq_i z)\right]\cos(kx)\,.$$

The integration constants C_T and C_S here, by virtue of the other equations of system (36), are related as

$$C_S = -\frac{\chi N_T^2}{\kappa N_S^2}\,C_T\,. \qquad (52)$$

That Eqn (52) must hold is apparent, for example, from the following considerations. Only if it is valid is the expression for the buoyancy $g(\alpha T - \beta S)$ free of the exponential $\exp(kz)$. Its absence is necessary because if it were present in the source (the last term) of dynamic equation in (36), it would also be present in the velocity field $\mathbf{v}$, which contradicts Eqn (49). The difference in the structure of solutions noted here for the fields of temperature and salinity on the one hand, and density, velocity, and pressure, on the other hand, makes them fundamentally different from the solution for a usual one-component fluid given, e.g., in Ref. [40].

The integration constants are found from the boundary conditions. Without repeating the rather cumbersome solution presented in Ref. [39], we here concentrate on the analysis of its features.

The solution depends on a number of dimensionless parameters, in particular, R, ξ, $b_T \equiv kh_T$, and $b_S \equiv kh_S$ (the last two are inverses of the respective analogs of Biot numbers [45]). Specific hydrodynamic properties of two-component media are traditionally associated with the difference in the exchange coefficients χ and κ (as described in Section 2). But in the problem considered here and in a number of other recent results, nontrivial effects were also found for $\delta = 1$. This corresponds, for example, to many geophysical models, which involve effective coefficients of turbulent exchange, approximately equal for all substances, temperature and salt included.

The depth dependence of the solution for velocity components (as well as for perturbations of pressure and medium density) is a linear combination of the three functions

$$\exp(2Kz),\ \exp(Kz)\cos(\sqrt{3}\,Kz)\,,\ \exp(Kz)\sin(\sqrt{3}\,Kz)\,, \quad (53)$$

where $K = k\mathrm{R}^{1/6}/2 \sim \mathrm{R}^{1/6}/L \gg L^{-1}$. The exponentials involved here decay with depth on the scale H of the order of $L/\mathrm{R}^{1/6}$, which in the approximation $\mathrm{R} \gg 1$ considered here is much shorter than the characteristic horizontal scale of perturbations $L \equiv 2\pi/k$. The wavelength of the sine functions above has the same order H. This result is an analog of the known solutions for usual one-component media [40, 44], in which perturbations look like a system of circulation cells arranged one under another, with the vertical and horizontal scales H and L and with their amplitude decreasing exponentially with the depth on the same characteristic scale of the order of H.

Thus, the circulation cells developing near the surface and the perturbation fields of velocity, pressure, and density are horizontally stretched, which justifies the above simplifications invoking the smallness of the ratio H/L. But the expressions for perturbations of temperature and composition in the case of two-component media, in addition to the above terms, generally contain one more term proportional to the exponential $\exp(kz)$ that decays relatively slowly. It has no analogs in one-component media and fundamentally changes the solution properties for emerging temperature perturbations.

As noted above, at first glance, the main effect of the admixture stratification should be a change in the density stratification and effective Rayleigh number (leaving aside specific effects of double diffusion related to the difference in the exchange coefficients of two substances). This explains why problems of this type are traditionally subject to the following simplifications [40]. By linearly combining equations of heat and admixture transport in the case of equal exchange coefficients, an equation for density (buoyancy) is derived that contains the net background density stratification. It is frequently thought that the problem is thus reduced to the known case of a one-component medium. However, for a variable such as density, it is generally impossible to correctly impose boundary conditions on the surface $z = 0$: the boundary conditions for the two components influencing density perturbations (temperature and admixture concentration) are usually different.

The solution in Ref. [39] demonstrates that the refusal to apply this unjustified simplification qualitatively changes the properties of solutions. The expressions for temperature and admixture concentration additionally acquire exponentials $\exp(kz)$ slowly decaying with depth. In other words, the perturbations of these substances, with all the other conditions being equal, can penetrate into two-component media (hydrostatically stable to an arbitrary degree) much more deeply than in one-component media, to depths of the order of horizontal scales L of inhomogeneous stresses. In the case of one-component media ($\gamma_S = 0$), the coefficient at the slowly decaying exponential in the solution obtained vanishes, as it does in the case of the same boundary conditions for the two substances ($h_T = h_S$). At depths exceeding H, the temperature and admixture concentration perturbations compensate each other to a substantial degree in the density field. This is why the perturbations of buoyancy, pressure, and velocity also penetrate into two-dimensional media only down to depths of the order of H, just as in the case of one-component media.

If a stable admixture stratification is added to the stable temperature stratification, it seems obvious that the perturbations should penetrate into the medium to a lesser degree because of the enhanced stability of density stratification if other conditions remain equal. This makes the result obtained even more nontrivial: the intuitive ideas notwithstanding, strengthening a stable stratification leads not to a reduced but to an increased penetration depth of thermal perturbation from values of the order of H to those of the order of L.

Upon strengthening a stable density stratification, the dimensionless parameter R (the effective Rayleigh number) also increases. Accordingly, the decrements of the exponentials $\exp(Kz)$ increase to some extent in absolute value, such that the respective terms in the solution discussed above decay faster with depth. This corresponds to a faster decay with depth of perturbations in velocity, density, pressure, and almost all terms in expressions for the temperature and admixture concentration. Nevertheless, the appearance of the additional exponential $\exp(kz)$ in the solution for T and S, which slowly decays with depth, can modify temperature profiles to a larger degree. Hence, despite the naive arguments listed above, the penetration depth of a perturbation in a two-component medium increases.

Figure 8 adapted from Ref. [39] presents examples of dimensionless vertical profiles of temperature in one- and two-component media. In the second case, the density stratification is more stable (the Rayleigh number R is twice as high). However, the perturbation in the stronger-stratified two-component medium has a larger amplitude and penetrates noticeably deeper.

The substantial difference between curves *2* and *1* in Fig. 8 owes its existence to the change in the background density stratification, which may seem insignificant at first glance. If the ratio of exchange coefficients is $\chi/\kappa = 10^{-2}$ (the case of seawater), the addition of a weak, stable stratification in admixture concentration to the stable temperature stratification in the numerical example above results in an increase in the background stratification by only 1%. It seems natural to expect that the temperature response to the mechanical action would be characterized by reduced amplitude and penetration depth, and that these changes would be small in absolute value. The result is notably different. In particular, the penetration depth of thermal perturbations increases by approximately one order of magnitude.

For definiteness, we consider the region of a positive tangent stress divergence (in the vicinity of the vertical axis $x = 0$), which corresponds to Figs 7 and 8. In this region, horizontally diverging flows arise near the surface. This implies (because of the continuity constraints) the generation of ascending vertical flows close to the surface [this is seen from both the schematics in Fig. 7 and the solution obtained (curve *3* in Fig. 8)]. Ascending motions carry colder and denser volumes of fluid from below, and hence the temperature in the surface layer decreases (see curves *1* and *2* in Fig. 8). In a one-component medium, the scales of this cooling (the amplitude and thermal effect penetration depth) are expressed by relations (44). They are determined by the balance between the external mechanical forcing causing a rise of the fluid in the domain considered and the negative buoyancy acquired by the rising fluid particles (relatively cold and dense).

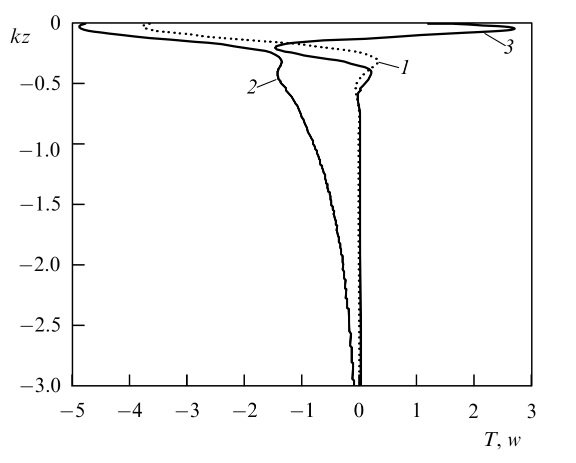

Figure 8. Examples of dimensionless profiles of temperature perturbations at the vertical line $x = 0$ for $\chi\alpha\gamma_T/\kappa\beta\gamma_S = -1$, $kh_T = 1$, and $kh_S = 0.05$ [39]. Curves *1* and *2* respectively pertain to one- and two-component media; curve *3* is the normalized profile of the vertical velocity w in the case of a two-component medium.

The situation becomes more involved in a two-component medium: the reduction in the fluid bulk temperature does not necessarily imply a reduction in its buoyancy. It also depends on the admixture concentration. Perturbations of these two substances can be mutually compensating in the buoyancy field, which implies the possibility of a reduced effect on the dynamics. Just this happens in the case above: in the solution obtained, an 'anomalously cold spot' near the surface turns out to also be 'anomalously fresh', such that the balance of the external mechanical action and deviations in buoyancy is not violated in the binary mixture, although relations (44) are violated. Because of the difference in boundary conditions for the two substances (if $h_S \neq h_T$), their sufficiently effective separation can occur in the system for the mechanical forcing of interest, giving rise to thermal effects of anomalous intensity.

Buoyancy forces in a medium affected by the field of gravity enforce the horizontal orientation of isopycnals (surfaces of equal density), with the exception of a thin surface layer of the thickness H, where this does not happen because of external mechanical forcing. In a usual one-component fluid, whose density is uniquely defined by its temperature, this also implies effective leveling of its isotherms — the absence of thermal perturbations outside the aforementioned thin layer. In binary mixtures that admit a separation of substances, the leveling of isopycnals does not prevent intense perturbations from occurring in the fields of each of the substances, which just happens in the example here.

The example above pertains to the case where both substances are stably stratified ($\gamma_T > 0$, $\gamma_S < 0$, and $\gamma_T/\gamma_S < 0$). But if we allow one of them to be stratified unstably ($\gamma_T/\gamma_S > 0$), then, for certain relations between the parameters, the amplitudes of perturbations can tend to infinity. This corresponds to a new mechanism of convective instability analyzed in Section 6.1.

5.3 Responses to thermal forcing

In analogy with mechanical stresses, we can also consider thermal inhomogeneities at the upper boundary [27, 46, 47]. The problem setup in this case differs from that in Section 5.1 only by the upper boundary conditions (the integration constants change accordingly). A nontrivial result is that the temperature can drop in the region of heat influx ('negative heat capacity'). The explanation is that the ascending motions driven by heating carry the colder volumes of fluid upward from below in the case of a stable thermal stratification. In one-component fluids, such a negative feedback can only partially compensate the heating. In two-component media, this feedback can be so strong that, counterintuitively, the transport of heat from below eventually leads to a reduction in temperature (it should be borne in mind that we are dealing with 'doubly nonequilibrium' media, stratified in temperature and admixture concentration.)

Similarly to the case of mechanical forcing, the addition of a stable saline stratification to a stable thermal one can result, rather counterintuitively, in an essential increase in the amplitude and penetration depth of thermal perturbations related to horizontal inhomogeneities of heat fluxes on the water surface. As in Section 5.2, this is explained by possible mutual compensation of contributions from two substances in the buoyancy field.

6. New mechanisms of convective instability

6.1 A new type of instability caused by double diffusion

Welander [48] proposed the idea of the possible existence of an unexplored type of convective instability in a two-component medium, notably, seawater, stratified in both temperature and admixture concentration (salinity). According to this hypothesis, a fluid with a stable density stratification can nevertheless lose its stability, not because of the difference in the heat and admixture concentration transfer coefficients (the known mechanism; see Section 2) but owing to the difference in boundary conditions at the horizontal boundary of the two-component medium. The feasibility of such an instability would be of considerable interest because, in contrast to the known mechanism (double diffusion), it could be realized in the presence of turbulent exchange when the effective transfer coefficients for heat and salt are practically equal, which, for example, is predominantly the case observed in the upper ocean layer. But this hypothesis was not proved in Ref. [48], where a grossly simplified theoretical system was considered (inviscid fluid, strongly idealized and strictly fixed boundary conditions, and so on). A sufficiently thorough analysis of linear instability for a semi-infinite domain was offered in Ref. [49]. It shows that the instability can indeed occur, but it differs essentially from that assumed in [48]. In particular, it is not oscillatory, as had been supposed, but monotonic, and unfolds for heating from above, not from below.

As in Section 5, a semi-infinite fluid layer $z \leqslant 0$ is considered (the z axis is directed vertically upward), stratified in temperature and admixture concentration (again, for definiteness, we assume seawater), and hence the hydrostatic equilibrium is stable (taken separately, thermal or saline stratifications can be unstable, but still give stable net density stratification).

The physical idea is as follows. For example, let a certain volume of a fluid with a stable thermal and unstable saline stratification be slightly displaced upward. Because the density stratification is stable, it seems that this volume (which is colder than the ambient medium) would attain negative buoyancy and experience a negative returning force. But its buoyancy also depends on the processes of exchange with the surrounding fluid. If the temperature at the horizontal fluid surface ($z = 0$) is fixed more rigidly than salinity (the boundary conditions for these two substances are different), the deviation of temperature in the displaced volume would relax faster than the deviation of salinity. Because the latter makes a positive contribution to the volume buoyancy and is more conservative than the negative temperature perturbation, a positive feedback is possible in principle. The mechanism is in certain respects analogous to the instability due to double diffusion [1, 2], but is linked to the difference in boundary conditions, in contrast to the difference in exchange coefficients.

As previously, we consider linearized system of equations (36). To explore whether the instability related to boundary effects can occur, we consider two-dimensional perturbations decaying far from the surface as $z \to -\infty$. We neglect fluid surface deformations, and hence the vertical velocity w vanishes at the surface $z = 0$. In addition to boundary conditions (38), it is also required that

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = 0\,. \tag{54}$$

The formulated stability problem is explored with respect to monotonic perturbations by using the standard normal mode method. The solution is sought in the form

$$w(x, z, t) = W(z)\cos(kz)\exp(\omega t) \tag{55}$$

(and similarly for other variables). Eliminating all unknowns except w from the original system of equations with $\omega = 0$ (bearing in mind computations of neutral curves), we arrive at Eqn (45). As mentioned, we are dealing with situations where the system is stable without the boundary effects taken into account. For example, an unstable saline stratification is more than compensated by the stable thermal stratification: $\gamma_T > 0$, $\gamma_S > 0$, $N_T^2 > 0$, $N_S^2 < 0$, $N_S^2 + N_T^2 > 0$, and $N_T^2/\kappa + N_S^2/\chi > 0$. The last inequality expresses one of the stability conditions on the known effects of double diffusion, capable of even destabilizing a medium with a stable density stratification [1, 2]. In agreement with the last condition, we consider only positive values of the parameter R.

We seek a solution of Eqn (45) in the form of a linear combination of exponentials. Under the condition of decay as $z \to -\infty$, a solution for the vertical velocity can include three exponentials. Expressions for the roots of the characteristic equation and the related stability analysis are rather cumbersome in general. But to demonstrate a new physical result (the existence of instability even for an arbitrarily strong stratification), it suffices to consider the asymptotic regime for $\mathrm{R} \gg 1$. In this case, the roots of the characteristic equation can be found approximately [see Eqns (51)], and the solution for the vertical velocity becomes

$$w \approx C_1 \exp(Kz) \times \left[\exp(Kz) - \cos(\sqrt{3}Kz) - \sqrt{3}\sin(\sqrt{3}Kz)\right]\cos(kx)\,,$$

$$K = \frac{k\mathrm{R}^{1/6}}{2}\,,$$

where C_1 is one of the constants of integration (the other two are found using boundary conditions for w and u). The dependence on depth of the solution for velocity components (and also for perturbations of pressure and medium density), as in Section 5.2, is given by a linear combination of the three functions in (53). They decay with depth on scales H of the order of $K^{-1} \sim L/\mathrm{R}^{1/6}$. In the approximation $\mathrm{R} \gg 1$ considered here, these scales are much smaller than the characteristic horizontal perturbation scale $L \equiv 2\pi/k$. The wavelength of sine functions at the last two exponentials is of the same order H. Solutions for perturbations of temperature and salinity additionally include the exponential $\exp(kz)$, which decays with depth much more slowly (on scales comparable to L):

$$T = \frac{\gamma_T}{\kappa k^2 \mathrm{R}^{1/3}} \Big\{ C_2 \exp(kz) + C_1 \exp(Kz) \big[\exp(Kz) + 2\cos(\sqrt{3}Kz) \big] \Big\} \cos(kx) ,$$
$$S = \frac{\gamma_S}{\chi k^2 \mathrm{R}^{1/3}} \Big\{ \frac{\chi \alpha \gamma_T}{\kappa \beta \gamma_S} C_2 \exp(kz) + C_1 \exp(Kz) \big[\exp(Kz) + 2\cos(\sqrt{3}Kz) \big] \Big\} \cos(kx) , \quad (56)$$

where C_2 is another integration constant. Using boundary conditions (38), we obtain a linear homogeneous system for C_1 and C_2. Its determinant passes through zero at the threshold of monotonic instability. We write the result: the instability domain is constrained by the inequality

$$\frac{\alpha \gamma_T / \kappa}{\beta \gamma_S / \chi} < \frac{(1 + kh_T)(\varepsilon + kh_S)}{(1 + kh_S)(\varepsilon + kh_T)} = \frac{1 + kh_T}{\varepsilon + kh_T} \, \frac{\varepsilon + kh_S}{1 + kh_S} , \quad (57)$$

where $\varepsilon = 3/2\mathrm{R}^{1/6}$ is a dimensionless parameter (small in the asymptotic case treated here). The dimensionless parameters kh_T and kh_S, as mentioned in Section 5. 2, are inverse to the respective analogs of the Biot numbers. The amplitudes C_1 and C_2 of the respective exponentials rapidly and slowly decaying with depth in expression (56) for the temperature perturbation are related as

$$C_2 = -2\mathrm{R}^{1/6} \frac{\varepsilon + kh_T}{1 + kh_T} C_1 = -\frac{3}{\varepsilon} \frac{\varepsilon + kh_T}{1 + kh_T} C_1 . \quad (58)$$

In particular, for $h_T = 0$ (no temperature perturbations at the boundary), $C_2 = -3C_1$. This relation is strongly dependent on boundary conditions: the ratio in the right-hand side of Eqn (58) shows substantial growth with h_T when the parameter kh_T exceeds the small parameter ε.

If the boundary conditions for both substances are identical ($h_T = h_S$), the right-hand side of inequality (57) is equal to unity. This corresponds to the known criterion of instability caused by double diffusion in a binary mixture [1]. But if the boundary conditions for heat and admixture are different, condition (57) can be substantially milder: the instability domain can then broaden such that the onset of instability becomes possible for an arbitrary stable density stratification (for $\mathrm{R} \to \infty$), even for equal heat and salt exchange coefficients.

We consider the dependence of the obtained criterion on the length scales h_T and h_S. The first fraction in the right-hand side of (57) depends only on h_T and reaches its maximum, equal to ε^{-1}, at $h_T = 0$. The second fraction depends only on h_S and increases monotonically with this parameter (tends to unity). Hence, the right-hand side of (57) reaches a maximum (tends to the large value ε^{-1}) at $h_T = 0$, $h_S \to \infty$. In other words, the limit case where one of the properties is subject to boundary conditions of the first kind (the surface temperature at $z = 0$ is strictly prescribed) and the other is subject to boundary conditions of the second kind, is most favorable for the onset of instability. A situation like this is discussed in Ref. [48], where the more general case of a condition of the third kind expressed by Eqn (38) was not considered. As can be seen from Fig. 9, the solution in the above limit is very sensitive to small changes in the boundary conditions (changes of the parameter $\delta \equiv h_T/h_S$).

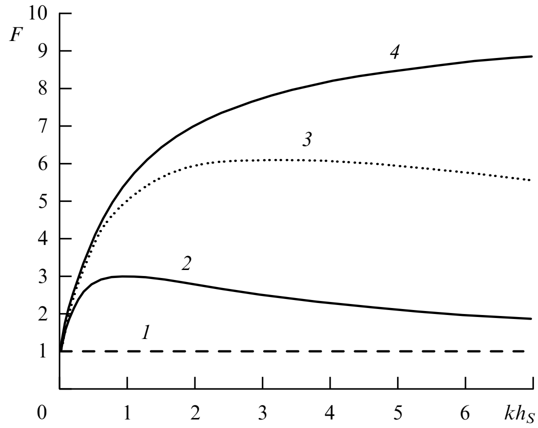

Figure 9. The right-hand side of inequality (57), $F = [(1 + kh_T)/(\varepsilon + kh_T)] \times [(\varepsilon + kh_S)/(1 + kh_S)]$, as a function of the dimensionless parameter kh_S for $\varepsilon = 0.1$ and various values of the ratio $\delta \equiv h_T/h_S$. Dashed line *1* corresponds to the same boundary conditions for the two substances ($\delta = 1$), curve *2* is for $\delta = 0.1$, curve *3* is for $\delta = 0.01$, and curve *4* for $\delta = 0$. The instability domains lie below the respective curves.

The most 'dangerous' mode occurs for $k = k_* = (\varepsilon / h_T h_S)^{1/2}$ because the right-hand side of (57) then reaches the maximum $[(1 + \sqrt{\varepsilon \delta})/(\sqrt{\varepsilon} + \sqrt{\delta})]^2$. The horizontal scale of this mode slightly exceeds the geometrical mean of the scales h_T and h_S, $k_*^{-1} = \mathrm{R}^{1/12}\sqrt{(2/3) h_T h_S}$. Figure 9 demonstrates a broadening of the instability domain as the difference between the boundary conditions for the two substances increases (the dimensionless ratio $\delta \equiv h_T/h_S$ is varied from one to zero). The abscissa corresponds to the dimensionless quantity kh_S. The region below line *1* corresponds to the known instability mechanism related to the double diffusion. As $kh_S \to \infty$, curve *4* asymptotically approaches the value ε^{-1}, as found above.

Figure 10 gives examples of vertical profiles of neutral perturbations for a rather stable density stratification (an analog of the Rayleigh number $\mathrm{R} = 3 \times 10^7$). A case is considered where the salinity introduces only a small destabilizing effect into the stable background density stratification (of the order of $\varepsilon \approx 0.1$) and the exchange

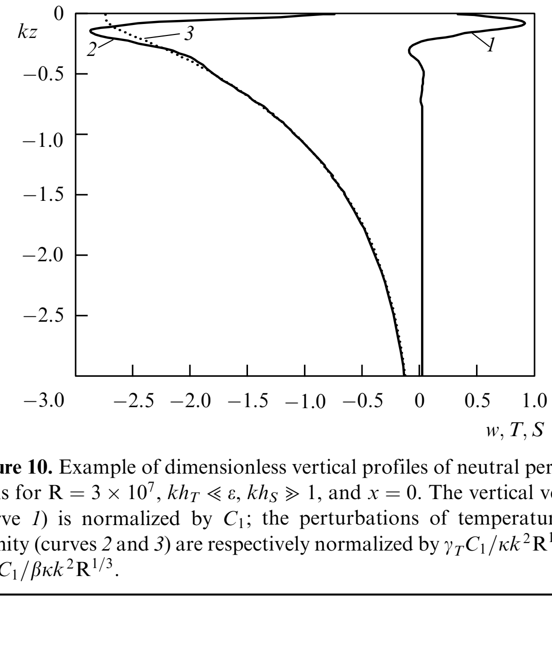

Figure 10. Example of dimensionless vertical profiles of neutral perturbations for $\mathrm{R} = 3 \times 10^7$, $kh_T \ll \varepsilon$, $kh_S \gg 1$, and $x = 0$. The vertical velocity (curve *1*) is normalized by C_1; the perturbations of temperature and salinity (curves *2* and *3*) are respectively normalized by $\gamma_T C_1 / \kappa k^2 \mathrm{R}^{1/3}$ and $\alpha \gamma_T C_1 / \beta \kappa k^2 \mathrm{R}^{1/3}$.

coefficients for both substances are equal (for example, when they are the effective coefficients of turbulent exchange). For example, the following parameter values correspond to this situation: $\alpha = 2 \times 10^{-4}$ K^{-1}, $\beta = 0.76 \times 10^{-3}$ (‰)$^{-1}$ [1 ‰ = 1000 ppm (parts per million)], $\gamma_T = 1.5$ K m^{-1}, $\gamma_S = 0.04$ (‰) m^{-1}, $\nu = \kappa = \chi = 10^{-3}$ m^2 s^{-1}, and $k = 0.1$ m^{-1}. The previously known instability mechanisms are not active in this case.

The profiles of temperature and salinity perturbations in Fig. 10 are normalized such that their ratio is equal to the ratio of their contributions to the perturbation of buoyancy. As can be seen from the figure, given the difference in boundary conditions for the perturbations of two tracers ($h_T \ll \varepsilon k^{-1}$, $h_S \gg k^{-1}$), the deviation of buoyancy close to the surface turns out to be positive, in agreement with the physical mechanisms described above. Accordingly, ascending motions should be generated near the surface (curve *1*). Below a layer with the thickness of the order of $H \sim K^{-1} \sim (k\mathrm{R}^{1/6})^{-1} \ll L$, the signs of buoyancy and velocity perturbations are reversed. Neutral perturbations in the velocity field form a vertical sequence of circulation cells rapidly decaying with depth. Perturbations in buoyancy and pressure decay similarly rapidly (on scales of the order of H). But the perturbations of temperature and salinity taken separately decay much more slowly with depth (on scales of the order of the horizontal wavelength L). Hence, although the discovered instability is linked with boundary effects, perturbations related to it can penetrate rather deep inside the fluid.

Previously known mechanisms of double diffusion, as mentioned in Section 2, can lead to the occurrence of an oscillatory instability in addition to the monotonic one (depending first and foremost on the relation between the background stratifications associated with the two substances). It cannot be excluded that an analog of a similar oscillatory instability domain also exists for the mechanisms considered here. Welander's hypothesis [48] is to a larger degree related to such a variant, which has not yet been rigorously explored.

Thus, the existence of previously unexplored types of convective instability of hydrostatically stable binary mixtures has been discovered recently. The instability of such media had previously been attributed exclusively to the essential difference in diffusion coefficients ($\chi \ll \kappa$). However, in [49], instead of the known condition $\chi\alpha\gamma_T/(\kappa\beta\gamma_S) < 1$, condition (57) was derived, which can be much less stringent in general. This implies that even for equal diffusion coefficients, a weak unstable stratification of one of the properties ($|N_S| \ll |N_T|$) can theoretically destabilize a medium with a stable density stratification.

The instability considered above stems from the difference in boundary conditions for two substances on the surface (the horizontal boundary). It is noteworthy that the horizontal placement of the boundary does not play a principal role. An instability of a similar character occurring at a vertical boundary in binary mixtures was analyzed in [50]. It was shown that even for a two-component medium with an arbitrarily strong static stability, its mechanical equilibrium in a gravitational field can nevertheless be unstable because of the difference in the boundary conditions for temperature and the admixture concentration at the vertical boundary. Analytic expressions for the criteria of instability of long-wave perturbations near a vertical boundary and in a vertical layer were also proposed there.

6.2 Anomalous thermocapillary instability in two-component media

Convective instability related to the thermocapillary effect (the Bénard–Marangoni convection) is usually considered only for rather thin layers of fluid (up to several millimeters in water). It is commonly believed that in thicker layers such an instability would be masked in the background of manifestations of the Rayleigh–Taylor instability [1, 3, 42, 51, 52]. However, in a two-component medium (for example, in seawater), situations can emerge where, for an unstable thermal stratification (heating from below), the density stratification is nevertheless statically stable owing to the stable stratification of the admixture concentration (salinity). The Rayleigh–Taylor instability is then suppressed, as is double-diffusive convection [1, 2], but the thermocapillary instability turns out to be not only possible but also free of 'competition'. Exploring such an instability mechanism in arbitrarily thick layers then becomes sensible.

This point was raised in Ref. [43], which offers an example of the stability analysis in a semi-infinite body of water stratified in both temperature and salinity. It was shown that despite the stable density stratification, a broad, previously unknown instability domain can exist owing to the thermocapillary effect.

As became apparent later, the set of boundary conditions for perturbations of the temperature and admixture concentration considered in Ref. [43] turned out to be related (to a degree, accidentally) to a rather special, degenerate case. This conclusion follows from the analysis of the results in Ref. [39] obtained several years later. As noted in Section 5, the response of stratified two-component media to the action of horizontally inhomogeneous tangent stresses at the upper boundary was studied in Ref. [39] theoretically. It was discovered that temperature perturbations with anomalous amplitude and penetration depth in a hydrostatically stable medium can increase due to the two-component medium composition. This allows assuming that anomalous thermocapillary effects can exist in such media, even in the presence of a stable density stratification. However, according to Ref. [39], the effects discovered there rely principally on the difference in boundary conditions for the temperature and admixture concentration perturbations. The analysis in [43] was limited to the case of similar boundary conditions (of the second kind). An important class of phenomena hinging on the difference in the boundary conditions was therefore overlooked. A more general case of different boundary conditions for each substance was analyzed in [53], where it was shown that this can indeed make the instability domain substantially wider and lead to qualitative diversity in the structure of growing perturbations.

We investigate the stability of a quiescent semi-infinite volume of a two-component medium (for definiteness, we consider water stratified in temperature and salinity), taking the thermocapillary effect at the upper horizontal boundary into account. The problem setup differs from that in Section 6.1 by one of the boundary conditions: instead of (54), we impose the condition commonly used to describe thermocapillary phenomena [3, 42]:

$$\rho_0 \nu \frac{\partial \mathbf{u}}{\partial z} = -\sigma_T \nabla_{\mathrm{h}} T \,, \qquad z = 0 \,. \tag{59}$$

Here, $\mathbf{u}$ is the vector of horizontal velocity, ∇_{h} is the horizontal Hamilton operator, and σ_T is the absolute value of the temperature derivative of the surface tension coefficient. In this section, our consideration is not confined to two-dimensional perturbations.

The instability of an unbounded layer of a two-component medium has already been explored in detail previously (see, e.g., Refs [1, 9]). Assuming that the values of the parameters γ_T and γ_S correspond to a stable state of the unbounded layer [9], we explore the possibility of an instability induced by surface effects. Accordingly, we consider perturbations decaying far from the surface, as $z \to -\infty$. As previously, deformations of the fluid surface are neglected. This corresponds to the condition $w|_{z=0} = 0$, where w is the vertical velocity component.

The formulated stability problem is explored with respect to monotonic perturbations using the standard method of normal modes. We seek a solution in the form

$$w(x, y, z, t) = W(z) \exp\left[\mathrm{i}(k_x x + k_y y) + \omega t\right]$$

(and similarly for other unknowns). Eliminating all unknowns except w from the original system, for $\omega = 0$ (having in mind computations of neutral curves), we arrive at Eqn (45), in which $k^2 = k_x^2 + k_y^2$.

We note that because the salt transfer coefficient χ in water is two orders of magnitude smaller than the thermal conductivity coefficient κ, the positive parameter R depends on the salinity stratification much more strongly than on the temperature stratification when all other conditions are equal.

Expression (46) can be rewritten as

$$\mathrm{R} = \frac{N_S^2}{\chi \nu k^4}\left(1 + \frac{\chi N_T^2}{\kappa N_S^2}\right) = \frac{N_S^2}{\chi \nu k^4}\left(1 - \frac{1}{\xi}\right) \approx \frac{N_S^2}{\chi \nu k^4}\,,$$
$$\xi \equiv \frac{\kappa}{\chi}\,\frac{\beta \gamma_S}{\alpha \gamma_T} = -\frac{\kappa}{\chi}\,\frac{N_S^2}{N_T^2}\,.$$

Because the stable background salinity stratification should at least compensate the unstable temperature stratification, it follows that $\beta\gamma_S \geqslant \alpha\gamma_T$. If, as mentioned, $\kappa \gg \chi$, then $\xi \gg 1$ (for saline water, $\xi \gtrsim 100$).

A solution of Eqn (45) is sought in the standard way as a linear combination of three exponentials:

$$W(z) = \sum_{i=1}^{3} C_i \exp(q_i k z)\,, \qquad q_1 = (1 + \mathrm{R}^{1/3})^{1/2}\,,$$
$$q_{2,3} = \left[1 + \mathrm{R}^{1/3} \exp\left(\pm\frac{2}{3}\,\pi \mathrm{i}\right)\right]^{1/2} = q \exp(\pm \mathrm{i}\varphi)\,, \qquad (60)$$
$$q = (1 - \mathrm{R}^{1/3} + \mathrm{R}^{2/3})^{1/4}\,, \qquad \varphi = \begin{cases} \eta\,, & 0 \leqslant \mathrm{R} \leqslant 8\,,\\ \dfrac{\pi}{2} + \eta\,, & \mathrm{R} > 8\,,\end{cases}$$
$$\eta = \frac{1}{2} \arctan \frac{\sqrt{3}\,\mathrm{R}^{1/3}/2}{1 - \mathrm{R}^{1/3}/2}\,, \qquad \mathrm{Re}\, q_i > 0\,.$$

Expressions for the temperature and salinity perturbation amplitudes contain the fourth exponential $\exp(kz)$ in the general case. It was inessential in Ref. [43] because for the class of boundary conditions considered there, the coefficients at this exponential vanishes. In the more general case considered here, terms with this exponential play an important role: for sufficiently stable stratifications, such that the parameter R is large, all other exponentials in the solution decay rapidly with depth (on a scale much smaller than $L = 2\pi/k$). Therefore, the presence of the fourth exponent in solutions qualitatively changes the structure of perturbations: it implies that neutral perturbations of thermocapillary origin can, technically, penetrate deeply ($|z| \sim L$) into the stably stratified medium.

The homogeneous system of equations for the four coefficients of the exponentials follows from the boundary conditions

$$\sum_{i=1}^{3} C_i = 0\,, \qquad \sum_{i=1}^{3}\left(q_i^2 - \frac{\mathrm{M}}{q_i^2 - 1}\right) C_i - \frac{\kappa k^2 \mathrm{M}}{\gamma_T}\, C_T = 0\,,$$
$$\sum_{i=1}^{3} \frac{b_T q_i + 1}{q_i^2 - 1}\, C_i + \frac{\kappa k^2 (b_T + 1)}{\gamma_T}\, C_T = 0\,, \qquad (61)$$
$$\sum_{i=1}^{3} \frac{b_S q_i + 1}{q_i^2 - 1}\, C_i + \frac{\chi k^2 \alpha (b_S + 1)}{\gamma_S \beta}\, C_T = 0\,,$$

where

$$\mathrm{M} = -\frac{\sigma_T \gamma_T}{\rho_0 \kappa \nu k^2}$$

is an analog of the Marangoni number.

The stability threshold corresponds to the vanishing of the determinant D of system (61). In [43], a relatively simple limit case corresponding to boundary conditions of the second kind $b_T = b_S = \infty$ ($b_T \equiv k h_T$, $b_S \equiv k h_S$) was explored analytically. The more general analysis in Ref. [53] relies on the results in Ref. [43] as a test example.

Performing analytic manipulations with the help of the software package Mathematica (see, e.g., Ref. [54]) allows obtaining an expression for the determinant of system (61) in the general analytic form and simplifying it to a certain extent. This expression, depending on the set of dimensionless parameters (R, M, ξ, b_T, and b_S), is unwieldy in general. We give it in the important particular case where the temperature and salinity fields obey respective boundary conditions of the second and first kind at the surface $z = 0$ ($b_T = \infty$, $b_S = 0$):

$$D = \frac{2\mathrm{i}q(q^2 + q_1^2 - 2qq_1\cos\varphi)\sin\varphi}{(q_1^2 - 1)(1 + q^4 - 2q^2\cos 2\varphi)}$$
$$\times \Big\{\mathrm{M}(\xi - 1) + q^2 - 1 + q_1(q_1 + q_1 q^2 - 2q^2\xi)$$
$$+ 2q\big\langle\left[q_1(1 + q^2) - \xi(q^2 + q_1^2)\right]\cos\varphi + q(1 - \xi q_1)\cos 2\varphi\big\rangle\Big\}\,. \qquad (62)$$

The stability threshold here corresponds to the vanishing of the expression in curly brackets (all other cases where expression (62) reaches zero or infinity are trivial). A convenient dimensionless instability criterion can be associated, for instance, with the positive parameter

$$J = \frac{\mathrm{M}}{\sqrt{\mathrm{R}}} = -\left(\frac{\chi}{\nu}\right)^{1/2} \frac{\sigma_T N_T^2}{\rho_0 \alpha g \kappa \left(N_S^2 + (\chi/\kappa) N_T^2\right)^{1/2}}$$
$$= -\left(\frac{\chi}{\nu}\right)^{1/2} \frac{\sigma_T N_T^2}{\rho_0 \alpha g \kappa N_S (1 - 1/\xi)^{1/2}}\,. \qquad (63)$$

For $\xi \gg 1$ (the case of seawater),

$$J \approx -\left(\frac{\chi}{\nu}\right)^{1/2} \frac{\sigma_T N_T^2}{\rho_0 \alpha g \kappa N_S}\,. \qquad (64)$$

If the instability criterion is rewritten in the form

$$J > J_* ,$$

where J_* is some critical value of the parameter J, then in conjunction with Eqn (63), this condition indicates the instability region in the plane of stratifications (γ_S, γ_T) or (N_S, N_T):

$$N_S < \left[-\frac{\chi}{\kappa} N_T^2 + \frac{\chi}{\nu} \left(\frac{\sigma_T}{\rho_0 \alpha g \kappa J_*} \right)^2 N_T^4 \right]^{1/2} \approx -\left(\frac{\chi}{\nu} \right)^{1/2} \frac{\sigma_T N_T^2}{\rho_0 \alpha g \kappa J_*}$$

(the last approximate equality relates to the limit $\xi \gg 1$).

For boundary conditions of the second kind ($b_T = b_S = \infty$), the value $J_* \approx 5.828$ was given in [43], and the most 'dangerous' mode occurs for $\mathrm{R} \approx 34$. Figure 11 displays the dependences $J(\mathrm{R})$ obtained by equating the determinant D to zero for various values of ξ, b_T, and b_S (the instability regions are above the neutral curves). Curves *1* and *2* are close to the respective curve in Ref. [43] because they correspond to large values of b_T and b_S. Their proximity confirms the weak dependence of solutions on ξ for the values of this parameter considered here.

For sufficiently large values of R, the quantities q and q_1 tend to $\mathrm{R}^{1/6}$ as $\varphi \to \pi/3$. For $1 \ll \mathrm{R}^{1/6} \ll \xi$, the expression in curly brackets in Eqn (62) vanishes if

$$\mathrm{M}(\xi - 1) \approx -2\mathrm{R}^{1/2}(\mathrm{R}^{1/6} - 2\xi) - \mathrm{R}^{1/2}\xi = \mathrm{R}^{1/2}(3\xi - 2\mathrm{R}^{1/6}) \quad (65)$$

(only terms with the highest powers of R are kept). Hence, it follows that for perturbations in the respective range of wavelengths,

$$\mathrm{M}\xi \approx 3\mathrm{R}^{1/2}\xi ,$$

and therefore the critical value of the parameter $J \equiv \mathrm{M}/\sqrt{\mathrm{R}}$ is close to three. This agrees with curve *3* in Fig. 11 (it slowly decreases as R increases, and, as is indicated by numerical calculations, passes through the value of three at $\mathrm{R} \sim 10^5$). The comparison of curve *3* with curves *1* and *2* shows that in the wavelength range presented in Fig. 11, the change in the salinity boundary condition (from the second kind to the first kind) widens the instability domain substantially, as could be expected according to the results in Ref. [39].

This result is strengthened even further if we consider perturbations with larger wavelengths (i.e., when R is larger than above). When R passes through the value $(3\xi/2)^6$, the right-hand side of Eqn (65) changes its sign. In other words, perturbations with such scales,

$$k < \left[\left(\frac{2}{3\xi} \right)^6 \frac{N_S^2}{\chi \nu} \right]^{1/4} , \quad (66)$$

are unstable even in the absence of the thermocapillary effect. This agrees with the computation results presented in Fig. 12: curve *1* intersects the abscissa at a point that corresponds to the estimate above. We note that the neutral curves can strongly depend on the boundary conditions. For example, curves *1* and *2* in Fig. 12 correspond to apparently close values of b_S, but the solutions are fundamentally different (curve *2* increases without intersecting the abscissa; the instability for such a boundary condition is only possible due to the thermocapillary effect).

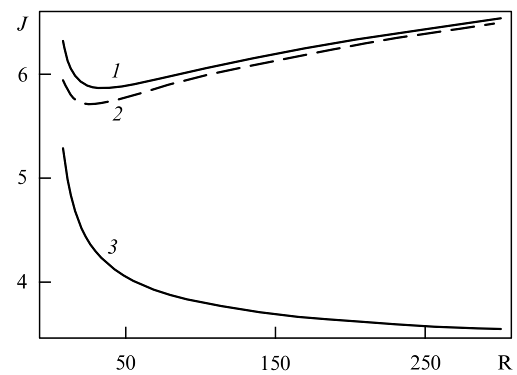

Figure 11. Neutral curves on the plane (R, J) for various boundary conditions and values of the parameter ξ. Curves *1* and *2*: $\xi = 80$, 150 and $b_T = b_S = 100$; curve *3*: $\xi = 80$, $b_T = 100$, and $b_S = 0.015$.

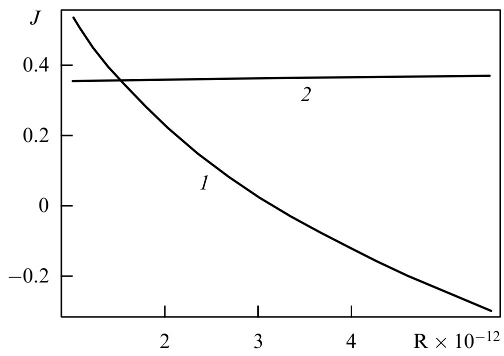

Figure 12. Neutral curves for large values of R, $\xi = 80$, and $b_T = 100$ for $b_S = 0$ (curve *1*) and $b_S = 0.015$ (curve *2*); curve *2* is normalized by 10.

The aforementioned possibility of the onset of instability in the absence of the thermocapillary effect corresponds to the situations considered in Section 6.1, where it was shown that an instability can arise because of the difference in boundary conditions for temperature and admixture concentration. In the case considered, this mechanism just amplifies the action of the thermocapillary effect, expanding the instability domain (and, in the long-wave limit, makes the occurrence of the instability theoretically possible even in the absence of the thermocapillary effect), whereas the better known feedback mechanisms (proposed by Rayleigh or linked to double diffusion), in contrast, stabilize the system.

We take values characteristic of seawater: $\rho_* = 10^3$ kg m^{-3}, $\sigma_T = 1.4 \times 10^{-4}$ N m^{-1} K^{-1}, $\nu = 10^{-6}$ m^2 s^{-1}, $\kappa = 1.4 \times 10^{-7}$ m^2 s^{-1}, $\chi = 1.5 \times 10^{-9}$ m^2 s^{-1}, $\alpha = 2 \times 10^{-4}$ K^{-1}, and $\beta = 0.76 \times 10^{-3}$ (‰)$^{-1}$. For $\gamma_S = 0.33$ ‰ cm^{-1}, we then obtain $N_S \approx 0.5$ s^{-1}; for the instability to develop in the case of homogeneous boundary conditions of the second kind for both substances, the unstable vertical background temperature gradient must be of the order of 1 K cm^{-1}. This corresponds to $\xi \sim 100$, while the horizontal scale of neutral perturbations and their depth of penetration into the fluid reach several millimeters by the order of magnitude. The terms with $\exp(kz)$ in the solution are not essential in this case.

The situation changes dramatically when boundary conditions for the two substances are different. For example, for $b_T = \infty$ and $b_S = 0$ (salinity perturbations satisfy the homogeneous boundary conditions of the first kind and temperature perturbations satisfy boundary conditions of the second kind), the solution for long-wave perturbations can be dominated by the term with $\exp(kz)$. This implies that such terms can have an anomalously high amplitude (compared to the case of one-component media) and penetrate deep into the medium despite its strong stable stratification. Such perturbations rather effectively bring about the positive feedback described in Refs [39, 49], which eventually leads to an increase in the amplitude and substantial expansion of the instability domain.

It should be borne in mind that long-wave perturbations, capable of penetrating deep into the medium, evolve only slowly. For example, for the parameters listed above, condition (66) corresponds to perturbations with wavelengths and penetration depths in excess of 1 m. Their development requires time intervals that are sufficient for diffusive transport of all properties over large spatial scales of the order of k^{-1}. Depending on concrete cases, such a slow evolution can in practice be equivalent to the absence of instability. And yet, formally, the instability domains can be rather wide.

In summary, with surface effects taken into account, a substantial part of the domain of physical parameters characterizing the upper layer of a two-component medium, traditionally considered as describing a stable state, is strictly speaking in the unstable range. The two-component character of the medium enables the development of intense perturbations that can penetrate deep into the medium despite its stability according to all known criteria.

6.3 Convective instability caused by a background flow

The instability caused by double diffusion, described in Section 2, arises in situations where the thermal stratification of the fluid is stable, but a slowly diffusing admixture (for example, salt in the case of seawater) contributes to the density stratification in a destabilizing manner. Although this contribution is small, it can be sufficient to destabilize a system stably stratified with respect to density because of the difference in exchange coefficients. The possibility of instabilities in qualitatively different situations was noted in [55], in particular, when a weakly diffusing property contributes, conversely, in a stabilizing way to the density stratification and this contribution can strongly exceed the 'thermal' instability in absolute value. At first glance, such a possibility looks even more paradoxical than the known mechanism described in Section 2 because, owing to the effects of double diffusion, the slowly diffusing property commonly affects the convective instability much more strongly (with other conditions being equal) than rapidly transferred heat does (the transfer coefficients enter the denominators of the respective analogs of the Rayleigh number).

Situations characterized by slow background motion in the direction of the gravity force are considered in [55]. Convection in the field of slow vertical motions is of interest, in particular, in relation to the known geophysical applications (see, e.g., Refs [56, 57]). Convection in the atmosphere and ocean often occurs in the background of motions with essentially larger horizontal scales (exemplified by atmospheric cyclones and anticyclones), which imply mean vertical motions that are several orders of magnitude slower than motions accompanying convective instabilities. A nontrivial fact is that according to field observations, even slow background downward motion effectively suppresses convection. The nature of the effects of that kind is little explored.

In [55], the following modification of the classical Rayleigh–Bénard problem of convective instability in fluids between two horizontal plates [1, 51] was considered. It is assumed that the background state is that of slow downward motion. For simplicity, it is supposed that the velocity of this motion, $-W < 0$ (W is the absolute value of velocity), is independent of the vertical coordinate z measured upward from the boundary at $z = 0$.[1]

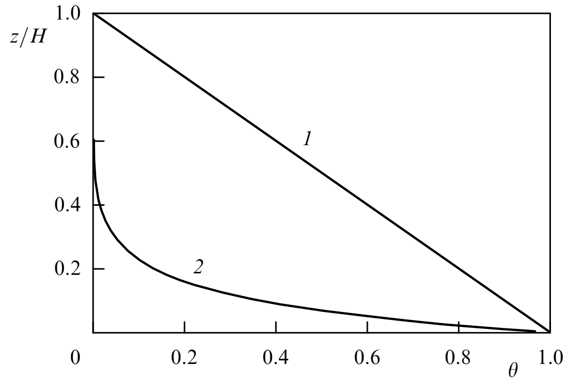

Figure 13. Example of the deformation of the background vertical temperature profile by descending fluid motions. Curve *1*: the vertical background motion is absent, curve *2*: the profile for $\varpi = H/h = 10$.

First, a one-component medium is considered, with the density dependent only on the temperature T (and with the effects from admixture stratification neglected). Temperatures T_d and T_u are fixed at the lower and upper boundaries, respectively, and their difference $T_d - T_u$ is denoted by ΔT. The heat transfer equation for the background state has the form

$$-W \frac{dT}{dz} = \kappa \frac{d^2 T}{dz^2} . \qquad (67)$$

Its solution with the above boundary conditions can be written as

$$\theta(z) = \frac{\exp(-\zeta) - \exp(-\varpi)}{1 - \exp(-\varpi)} , \qquad (68)$$

where $\theta(z) = (T(z) - T_u)/\Delta T$ is the dimensionless temperature deviation, $\zeta = z/h$ is the dimensionless height, $h = \kappa/W$ is the height scale associated with the background vertical motion (in its absence, h tends to infinity), $\varpi = H/h = W(H/\kappa)$ is the most important dimensionless parameter of the problem, and H is the layer thickness. In the absence of vertical background motion (in the limit $W \to 0$, $h \to \infty$, $\varpi \to 0$), as expected, we obtain the linear profile $\theta = 1 - z/H$, whose stability is analyzed in the standard Rayleigh problem.

Figure 13 presents vertical profiles $\theta(z)$ for the dimensionless parameter ϖ equal to 1 and 10. It can be readily seen that the background descending motion 'pushes' practically all the vertical temperature contrast ΔT to the lower boundary, such that it occupies a layer with a thickness of the order of $h = \kappa/W$. A rigorous stability analysis of a stationary state with a curved temperature profile and background descend-

[1] The assumption of a constant vertical velocity implies that the impermeability condition is no longer imposed at the horizontal boundaries $z = 0, H$ for the background motion. Such configurations can be practically implemented by pumping fluid in the vertical direction through porous horizontal boundaries. This assumption essentially simplifies the computations.

ing motion is a rather tedious and cumbersome task. But physical considerations allow making a simple and rather plausible estimate. It seems quite obvious that the stability of state *2* in Fig. 13 is determined by the convective stability of the lower 'sublayer' with a thickness of the order of $h = \kappa/W$, which roughly confines the entire temperature contrast ΔT. The effective Rayleigh number for such a sublayer is

$$\mathrm{Ra} \sim \frac{\alpha g \Delta T h^3}{\kappa \nu} \sim \frac{\alpha g \Delta T \kappa^2}{\nu W^3}\,, \tag{69}$$

where α is the medium thermal expansion coefficient.

According to (69), the effective Rayleigh number strongly depends on the vertical background motions. This is natural because they determine the thickness of the sublayer containing practically the entire temperature gradient. We also note that expression (69) rapidly increases with the thermal conductivity coefficient κ (the classical Rayleigh number, in contrast, decreases as κ increases). We let $\mathrm{Ra}_{\mathrm{cr}}$ denote the value of the effective Rayleigh number that corresponds to the stability loss. In this case, the downward velocity sufficient to suppress the onset of convective instability is expressed as

$$W_{\mathrm{cr}} \sim \left(\frac{\alpha g \Delta T \kappa^2}{\nu \mathrm{Ra}_{\mathrm{cr}}}\right)^{1/3}. \tag{70}$$

For example, let $\kappa = \nu = 1\ \mathrm{m^2\ s^{-1}}$ (the effective values of turbulent exchange coefficients characteristic for the atmospheric boundary layer), $\alpha = 4 \times 10^{-3}\ \mathrm{K^{-1}}$, $\Delta T = 0.1$ K, and $\mathrm{Ra}_{\mathrm{cr}} = 10^3$; then $W_{\mathrm{cr}} \sim 10^{-2}\ \mathrm{m\ s^{-1}}$. This descending motion is two to three orders of magnitude weaker than characteristic convective velocities in the atmosphere. Nevertheless, according to field measurements and numerical simulations, such motions indeed effectively suppress convection.

If stratification in a slowly diffusing admixture (salt) is added to the temperature stratification, it is 'blown off' by the background motions more efficiently than heat. For example, in the absence of descending background motions, let a stable salinity stratification stabilize the system with an unstable temperature stratification. Then the presence of even slow descending motions can lead to the situation where an admixture (different from heat) is blown off and does not act as a stabilizing factor; the system becomes unstable. This situation was analyzed in detail in [55].

We reiterate that background vertical motions in certain situations can suppress the onset of convective instabilities, but in two-component media, by contrast, they can destabilize the fluid layer. This happens because the less diffusive substance is blown off by background motions more efficiently. It is noteworthy that these effects are generally possible even for very small vertical velocities.

6.4 Instability caused by a faster transport of one of the components

Paper [58] drew attention to one more mechanism of convective instability in two-component media. At the upper surface of such a medium (for example, saline water), let sinks of salt and heat start to act simultaneously, such that the net buoyancy source at the upper boundary is positive (i.e., the source of buoyancy related to water freshening is stronger than that related to its cooling). At first glance, such a source of positive buoyancy from above can only stabilize the medium. But because heat diffuses much faster than salt, the temperature perturbation propagates down faster than the saline one, and their mechanical effect becomes spatially separated: it becomes applied to different regions. As follows from estimates, even a weak temperature perturbation, which nonetheless propagates farther from the boundary, can trigger convective instability in the medium, despite being followed by a strong stabilizing perturbation of salinity. A nontrivial aspect is that this effect is possible even when a net stabilizing buoyancy flux operates in the background of a stable state of the medium.

Several years later, an effect of that type was experimentally discovered in Ref. [59], whose author was most likely unaware of Ref. [58]. As mentioned in Ref. [59], underneath a floating piece of ice, ''because of large difference in coefficients of molecular diffusion the thickness of thermal boundary layer grows faster than the thickness of concentration layer. As a result, a part of solution is cooled preserving the initial salinity, becomes denser and sinks to larger depths....'' Paper [60], published practically simultaneously with [59], reports on similar sinking motions observed under an iceberg in the Barents Sea. The mechanism described above was apparently unknown to the authors of Ref. [60] (they hypothesized instead that the ice carried solid inclusions of bedrock, creating an effect of negative buoyancy).

6.5 Convective–radiative instability of moist air

A specific example of a binary mixture is furnished by unsaturated moist air, whose density depends on both the temperature and water vapor concentration. The air density decreases if the concentration of water vapor or temperature increases.[2] Lower layers of the atmosphere are characterized by situations where temperature stratification is stable, but humidity contributes to it in a destabilizing manner. In other words, the background vertical density gradients associated with the two substances oppose each other. According to the material presented, for example, in Sections 2 and 6.1, just such situations are conducive to new interesting effects (loss of stability, 'negative heat capacity,' and others). Reference [63] draws attention to the theoretical possibility of a convective instability to occur (despite the stable density stratification) in this particular case. The point is that the perturbations of temperature and water vapor concentration relax at different rates in general. Heat exchange involves diffusion and radiative effects, while the exchange in moisture involves only diffusion. Thus, a mechanism exists that qualitatively resembles double diffusion in saline water, described in Section 2. Estimates demonstrating that such an instability mechanism is feasible in principle are given in [63].

6.6 Instability linked to phase transitions at a boundary

A previously unknown mechanism of convective instability of unsaturated air over a wet surface was discovered and explored theoretically in [64, 65]. The physical idea can be elucidated as follows. We consider a medium (semi-infinite in the simplest case) stratified as in Section 6.5: the temperature increases with height, while the waver vapor concentration decreases, and the net stratification is statically stable. Let a perturbation in the form of slow descending motion be excited near the lower boundary $z = 0$ of some surface region. It carries dry air parcels to the surface. For a wet

[2] More precisely, in the case of air, instead of temperature, we are dealing with the potential temperature [61, 62]. It is a more convenient variable because it automatically accounts for the effect of air compressibility.

surface, the arrival of dry air implies enhanced surface evaporation, i.e., some additional cooling for the respective region, as well as for the adjacent air. A 'cold spot' forms in this way in the region of descending motions. A related negative buoyancy in the vicinity of the region being cooled further amplifies the downward motions: we can speak about a positive feedback. Admittedly, there are negative feedbacks as well in the process considered. For example, the descent of a medium with a stable temperature stratification leads to an increase in temperature and buoyancy, which acts to suppress the downward motion. Nevertheless, a thorough analysis of the stability problem performed in Refs [64, 65] points to the existence of an instability domain that is sufficiently broad and important for applications.

Additional feedback appears if there is a vertically adjacent ($z < 0$) stably stratified layer of water (the temperature in water decreases with depth). Indeed, the aforementioned descending motion of air also implies its divergent flow near the water surface, $z = 0$. Because of tangent stresses at the air–water interface, the upper layer of water also becomes involved to some extent in this motion. From the continuity, it follows that the leaving horizontal water parcels have to be replaced by parcels coming from below. The water coming from below (which is colder) additionally amplifies the surface cooling discussed above. This positive feedback can be significant, even in the absence of evaporation [41], and even more effective when evaporation is present. A related joint instability problem is analyzed in detail in Refs [41, 66, 67].

7. Effects of rotation

Thus far, the main role has been played by effects of stratification in density and each of the substances influencing it. In geophysical fluid dynamics, the effects of rotation are as significant as the stratification [61, 62]. Slow motions in the atmosphere and ocean are characterized by the state of the so-called geostrophic balance, in which the horizontal gradient of pressure is balanced by the Coriolis force in the frame of reference corotating with the earth. Deviations from this balance launch a wave process of geostrophic adjustment, by virtue of which the velocity and pressure fields evolve, gradually returning to a new geostrophically balanced state. This process plays a rather important role in the dynamics of the atmosphere and ocean; in particular, it is an important source of wave perturbations. Its analysis is the subject of vast literature [61, 62], but its specific features in two-component media (for example, when temperature and salinity stratifications have to be taken into account simultaneously) remained practically unexplored until recently. This gap is filled by recent papers [68–70].

Accounting for rotation essentially broadens classes of nontrivial effects in the hydrodynamics of two-component media. Their detailed description is beyond the scope of this review. In addition to the already mentioned effects of 'negative heat capacity', amplification of thermal perturbations in a medium stable according to all previously known criteria, and hydrodynamic 'memory' of binary mixtures, it is worth separately mentioning the possibility of formation of discontinuities (jumps) in distributions of both substances that determine the fluid density.

Earlier, one of us showed [70] that discontinuities can form in the process of nonlinear geostrophic adjustment in usual one-component media if there are sufficiently high amplitudes of initial perturbations. This effect underlies the theory of formation of atmospheric and oceanic fronts proposed in [28, 70]. Accounting for the two-component character of a fluid makes the qualitative properties of discontinuous surfaces particularly diverse. For example, a frontal surface can be distinctly expressed in the field of one of the thermodynamic components (temperature) and weakly in the other (salinity). Features of this kind show up in the observed distributions of hydrodynamic fields in the ocean.

We consider the specific features of the dynamics of large-scale (quasi-geostrophic) motions in stratified rotating two-component media in more detail [71]. In the framework of geophysical fluid dynamics, these are motions characterized by a small Rossby number $\mathrm{Ro} = U_0/fL$, where U_0 and L are the respective typical values of velocity and horizontal scale, and f is the Coriolis parameter (the projection of the doubled angular velocity vector on the local vertical). In geophysical applications, it typically happens that the dependence of the Coriolis parameter on latitude (meridional coordinate y) is essential for large-scale motions. The frequent practice is to invoke the so-called beta-plane approximation, in which $f \approx f_0 + \beta_* y$, where f_0 and β_* are constant. For quasi-geostrophic motions, the horizontal velocity $\mathbf{v} = (u, v)$ and buoyancy perturbations σ are found from the relations of geostrophic and hydrostatic balance [61, 62]

$$\mathbf{v} = \mathbf{v}_\mathrm{g} = \mathbf{k} \times \nabla_\mathrm{h}\psi \,, \qquad \sigma = \frac{f_0}{g}\frac{\partial\psi}{\partial z} \,, \tag{71}$$

where $\psi = p/f_0\rho_*$ is the geostrophic streamfunction (proportional to the pressure perturbation) and ∇_h is the horizontal gradient operator. The streamfunction evolution is in this case governed by the quasi-geostrophic potential vorticity equation (sometimes called the Charney–Obukhov equation)

$$\frac{Dq}{Dt} = 0 \,, \qquad q = f_0 + \beta_* y + \Delta_\mathrm{h}\psi + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z} \,, \tag{72}$$

where N^2 is the buoyancy frequency squared and D/Dt is the geostrophic derivative operator,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v}_\mathrm{g}, \nabla_\mathrm{h}) = \frac{\partial}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} \,.$$

The standard derivation of Eqn (72) is based on an asymptotic expansion in power series in Ro and is well explained in tutorial courses and books [61, 62]. Equation (72) lies at the heart of the so-called quasi-geostrophic model traditionally invoked in exploring the dynamics of large-scale Rossby waves (caused by the 'beta-effect' — the dependence of the Coriolis parameter on the meridional coordinate) and synoptic eddies in the atmosphere and ocean.

Solving Eqn (72), we find the geostrophic streamfunction (pressure perturbation) and then, from the geostrophic and hydrostatic balance in (71), the fields of velocity and density (buoyancy). The question about temperature and salinity fields remains beyond the scope of the standard quasi-geostrophic model; in other words, Eqn (72) alone is insufficient for an unambiguous description of a two-component medium. To obtain the missing equation, we write the transport equations for heat and salinity (3) in the quasi-geostrophic approximation, neglecting vertical nonlinear advection and approximating the full derivatives by the operator D/Dt:

$$\frac{DT}{Dt} + \gamma_T w = 0\,, \qquad \frac{DS}{Dt} + \gamma_S w = 0\,. \tag{73}$$

Here, T and S are the deviations from the background distributions (primes are omitted). Elimination of the vertical velocity from Eqns (73) leads to a conservation law for the invariant,

$$\frac{Dr}{Dt} = 0\,, \qquad r = \gamma_S T - \gamma_T S\,. \tag{74}$$

As with the quasi-geostrophic potential vorticity q, the invariant r is conserved by nonlinear quasi-geostrophic dynamics, i.e., it is a Lagrangian invariant. In contrast to the quasi-geostrophic potential vorticity, it plays the role of a passive tracer carried by the geostrophic velocity field. Obviously, Eqn (9) is a local form of Eqn (74).

The temperature and salinity fields can be uniquely determined from the distributions of the buoyancy σ and invariant r found from Eqns (72) and (74) (see Section 3). The respective contributions can be divided into the density-related and compensated components, and are expressed by formulas (12) and (13) obtained above. The full system of quasi-geostrophic equations, with the distributions of temperature and salinity included into the set of field variables, therefore consists of two equations (72) and (74) and simple algebraic relations (12). They form a closed model for a quasi-geostrophic dynamics of two-component media.

This model can be used to describe some nontrivial features of large-scale dynamics in two-component media. We first consider several exact solutions of the system formed by Eqns (72) and (74). Let the streamfunction $\psi = \bar{\psi}(y,z)$ be a simple stationary solution of Eqn (72) describing the zonal flow with the velocity $\bar{U}(y,z) = -\partial\bar{\psi}/\partial y$ and buoyancy distribution $\bar{\sigma} = (f_0/g)\,\partial\bar{\psi}/\partial z$. For this streamfunction, Eqn (74) reduces to the linear transport equation $\partial r/\partial t + \bar{U}(y,z)\,\partial r/\partial x = 0$, whose solution with the initial condition $r|_{t=0} = r_0(x,y,z)$ takes the form

$$r = r_0\big(x - \bar{U}(y,z)t, y, z\big)\,.$$

Its important feature is the unlimited growth of spatial derivatives with time:

$$\frac{\partial r}{\partial y} = \frac{\partial r_0}{\partial y} - t\,\frac{\partial \bar{U}}{\partial y}\,\frac{\partial r_0}{\partial x}\,, \qquad \frac{\partial r}{\partial z} = \frac{\partial r_0}{\partial z} - t\,\frac{\partial \bar{U}}{\partial z}\,\frac{\partial r_0}{\partial x}\,.$$

By virtue of relations (12), this growth also pertains to the derivatives of distributions T_r and S_r. The evolution of compensated distributions of temperature and salinity is inevitably accompanied by a sharpening of spatial gradients. This behavior may be one of the reasons explaining the formation of compensated thermohaline fronts in the ocean. Such fronts are manifested only as jumps in profiles of temperature and salinity distributions [62].

Interesting aspects of two-component media are manifested in the behavior of Rossby waves generated by an initial temperature perturbation in a moving fluid. Here, we explore the behavior of small perturbations of a zonal flow characterized by a uniform velocity $\bar{U} = -\partial\bar{\psi}/\partial y$. Assuming that $\psi = \bar{\psi} + \psi'$ and $r = r'$, we write Eqns (72) and (74) in the linearized form (primes are omitted in what follows)

$$\left(\frac{\partial}{\partial t} + \bar{U}\,\frac{\partial}{\partial x}\right)\left(\Delta_{\rm h}\psi + \frac{\partial}{\partial z}\,\frac{f_0^2}{N^2}\,\frac{\partial\psi}{\partial z}\right) + \beta_*\,\frac{\partial\psi}{\partial x} = 0\,, \tag{75}$$

$$\frac{\partial r}{\partial t} + \bar{U}\,\frac{\partial r}{\partial x} = 0\,. \tag{76}$$

Equation (75) is supplemented with the boundary conditions

$$\left(\frac{\partial}{\partial t} + \bar{U}\,\frac{\partial}{\partial x}\right)\frac{\partial\psi}{\partial z} = 0\,, \qquad z = 0, H\,,$$

which follow from impermeability of rigid boundaries $z = 0, H$ [61, 62]. An initial value problem is considered for Eqns (75) and (76) with the initial conditions at $t = 0$ given by

$$\psi = -\varepsilon_\psi A(x)\cos(kx)\cos(\lambda_n z)\,, \qquad r = \frac{\gamma_S}{\alpha}\,\sigma = \frac{\gamma_S f_0}{\alpha g}\,\frac{\partial\psi}{\partial z}\,,$$

where ε_ψ is a small amplitude parameter and $\lambda_n = \pi n/H$. With Eqn (12), these conditions correspond to the situation where only a perturbation of temperature exists initially in the form of a modulated wave packet, whereas the perturbation of salinity is absent, $S = 0$ and $T = \varepsilon_T A(x)\cos(kx)\sin(\lambda_n z)$ at $t = 0$. Here, $\varepsilon_T = (f_0\lambda_n/\alpha g)\varepsilon_\psi$ and $A(x)$ is the envelope slowly varying over the wavelength.

The asymptotic solution of Eqn (75) describing the propagation of a Rossby wave packet can be written as

$$\psi = -\varepsilon_\psi A(x - c_{\rm g}t)\cos\big[k(x - ct)\big]\cos(\lambda_n z)\,,$$

where $c_{\rm g} = \partial\omega/\partial k$ and $c = \omega/k$ are the respective group and phase velocities,

$$c_{\rm g} = \bar{U} + \frac{\beta_*(k^2 - l^2 - L_{\rm R}^{-2})}{(k^2 + l^2 + L_{\rm R}^{-2})^2}\,, \qquad c = \bar{U} - \frac{\beta_*}{k^2 + l^2 + L_{\rm R}^{-2}}\,,$$

$L_{\rm R} = N/f_0\lambda_n$ is the so-called baroclinic Rossby deformation radius [61, 62], ω is the frequency accounting for the Doppler shift, and l is the wave vector component along the y axis (for two-dimensional perturbations, $l = 0$). The solution of Eqn (76) has the form

$$r = \varepsilon_T\gamma_S A(x - \bar{U}t)\cos\big[k(x - \bar{U}t)\big]\sin(\lambda_n z)\,.$$

Determining buoyancy from Eqns (71) and using relations (12), we find the perturbations of temperature and salinity at the level $z = H/2$ (the first baroclinic mode)

$$\begin{aligned} T &= \frac{\varepsilon_T}{1-\eta}\Big\{A(x - c_{\rm g}t)\cos\big[k(x - ct)\big] \\ &\quad - \eta A(x - \bar{U}t)\cos\big[k(x - \bar{U}t)\big]\Big\}\,, \\ S &= \varepsilon_T\,\frac{\alpha}{\beta}\,\frac{\eta}{1-\eta}\Big\{A(x - c_{\rm g}t)\cos\big[k(x - ct)\big] \\ &\quad - A(x - \bar{U}t)\cos\big[k(x - \bar{U}t)\big]\Big\}\,, \end{aligned} \tag{77}$$

where the parameter η is defined in (5).

The first term in the curly brackets in (77) corresponds to the density-related component of the thermohaline perturbation, and the second to the compensated component. An important feature of this solution is that the envelope of the density-related component propagates with the group velocity of Rossby waves, whereas the compensated component propagates with the speed of the flow. Because of the specific

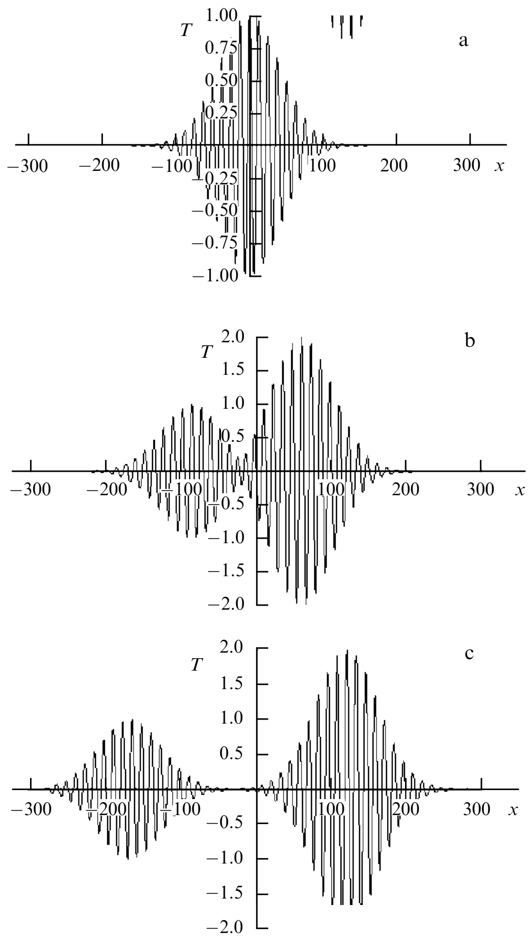

Figure 14. Temperature distribution (44) for time instants (a) $t = 0$, (b) $t = 60$, and (c) $t = 120$. The values of x, t, and T are respectively normalized by L_R, $L_R/\bar{U}$, and ε_T.

form of the dispersion relation for Rossby waves, not only the amplitudes but also the signs of these velocities can differ. Thus, for a westerly flow, $\bar{U} > 0$ (the abscissa points eastward, as is commonly assumed in geophysical fluid dynamics), the group velocity is westward if $k < k_*$, where $k_* \approx L_R^{-1}(1 - 4\bar{U}/U_R)^{1/2}$ for $\bar{U} \ll U_R = \beta_* L_R^2$.

Plots of temperature distribution given by expression (77) are shown in Fig. 14 for three time instants. The shape of the envelope was defined as $A(x) = \exp[-(x/\Delta)^2]$ and computations were performed for the following set of parameters: $\beta_* = 2 \times 10^{-8}$ km^{-1} s^{-1}, $L_R = 50$ km (the first baroclinic mode), $\bar{U} = 1$ cm s^{-1}, $U_R = 5$ cm s^{-1}, $kL_R = 0.5$, $\eta = 2$, and $\Delta \gg k^{-1}$. The phase and group velocities in this case are westward, $c = -3$ cm s^{-1} and $c_g = -1.4$ cm s^{-1}. The plots explicitly illustrate the break-up of the initial packet into the density-related and compensated components propagating in opposite directions. The amplitude of the compensated part can significantly exceed the initial amplitude ε_T. We stress that the picture obtained differs radically from that in one-component media, where perturbations are only density-related and propagate westward along with the Rossby waves.

8. Density currents caused by double diffusion

The presence of horizontal density gradients in a fluid subject to the field of gravity is responsible for flows called gravity currents [1, 18]. The nature of such flows is obvious — the difference in weights of two neighboring fluid columns implies a difference in the hydrostatic pressures at their base, i.e., the presence of horizontal pressure gradients, which drive horizontal flows. Dissipative factors commonly play a negative role in the development of these currents; they (even discarding viscous friction) typically smooth the density field gradients, thus reducing the main force driving the gravity currents.

It was noted in [72] that situations can occur where the same dissipative processes, on the contrary, induce gravity currents in a hydrostatically equilibrated medium that is stable according to all known criteria. Such situations can unfold in binary mixtures whose density depends on both temperature and admixture concentration. In this case, the difference in exchange coefficients for the two tracers is essential.

In Section 3, we described the memory effect in two-component ideal media. Thermal inhomogeneities (as well as inhomogeneities in the admixture concentration) created in such media do not necessarily disappear in the process of hydrostatic adjustment, but leave a long-lived trace such that the temperature and admixture concentration perturbations make no net contribution to the density field. In an ordinary one-component medium, the existence of such memory is certainly impossible: the horizontal alignment of isotherms in the field of gravity is always recovered in the process of hydrostatic adjustment, even in the absence of dissipative factors. The emergence of long-lived thermohaline traces was mentioned previously in laboratory experiments (see, e.g., Ref. [21, p. 221]) and in field observations in the upper ocean layer [34–36].

Therefore, there exist simple mechanisms that maintain the generation and long-term existence of horizontal thermal inhomogeneities in seawater that do not violate the horizontal density distribution, i.e., are hydrostatically stable. With the heat and mass exchange taken into account, thermohaline formations of this type gradually dissipate. The study of such processes was begun in [73]. However, rather special classes of perturbations in an infinite medium, independent of the vertical coordinate z, were considered in [73]. In a problem with that type of symmetry, it is possible to advance beyond the framework of small perturbation, but at the expense of losing horizontal motions (since $\partial/\partial z \equiv 0$, the continuity equation imposes the constraint of no horizontal velocity).

The case with a more general geometry was considered in [72] in the linear approximation. Because heat and the admixture diffuse at different rates, their mutual compensation in the density field gradually becomes incomplete with time. As a consequence, perturbations of density and pressure (because of hydrostatics) are generated, giving birth to density currents, usually slow on the hydrostatic adjustment time scale. Certain properties of such density currents, important as mechanisms dissipating the 'compensated' inhomogeneities widely spread in nature, are theoretically explored in Refs [72, 73].

9. Anomalous hydrodynamic drag

In [74], another effect was discussed, which seems to have not been considered previously: the possibility of an essential modification of hydrodynamic drag through the mechanism of double diffusion, even in media with a homogeneous density.

The physical idea is as follows. Let a two-component fluid (for example, saline water) be vertically stratified with respect to both substances (temperature and salinity) in the field of gravity. Let the density stratification be either neutral or stable. A localized vertical force (a source of momentum) applied to such a fluid would generate a vertical flow that perturbs both components in some vicinity of the applied force. Because of the difference in the exchange coefficients, the relaxation of these perturbations proceeds differently for temperature and salinity. This affects the relation between their contributions to the fluid density, i.e., the incurring buoyancy perturbations. They in turn modify the vertical motion relative to the region of applied force, which implies a change in the effective hydrodynamic resistance in the medium.

In [74], this was illustrated with a particular example where, owing to the high symmetry of the problem, it was possible obtain an exact analytic solution of the system of equations of hydrodynamics and admixture (salinity) transport in the Boussinesq approximation. The effect of a stationary, vertically homogeneous momentum source in an infinite medium stratified with respect to both temperature and admixture (salinity) was considered. In other words, we can speak of a homogeneously distributed force directed upward and applied along the z axis. The background density stratification is assumed to be either neutral or stable, excluding the convectively unstable cases. To describe perturbations generated by such a vertically distributed source, a cylindrical coordinate system is used, with its vertical axis z pointing upward and aligned with the source. A stationary solution is sought that, like the force, is independent of the vertical coordinate z. In this approximation, stationary perturbations appearing in the vicinity of the source depend only on the radial coordinate r. It then follows from the continuity equation (the Boussinesq approximation is used) that horizontal motions are absent. A detailed analysis pertains to the situation where the vertical force exerted on the fluid is confined to and homogeneously distributed along the z axis. In this case, in the approximation considered, the system of equations of hydrodynamics and heat and admixture transport takes the form [74]

$$\nu\left(\frac{\mathrm{d}^2 w}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right)+g(\alpha\theta-\beta S)=-P\,\frac{\delta(r)}{2\pi r}\,, \tag{78}$$

$$\kappa\left(\frac{\mathrm{d}^2 \theta}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)-\gamma_T w=0\,, \tag{79}$$

$$\chi\left(\frac{\mathrm{d}^2 S}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}S}{\mathrm{d}r}\right)-\gamma_S w=0\,, \tag{80}$$

where $\delta(r)$ is Dirac's delta-function. The constant $P\,[\mathrm{m^3\,s^{-2}}]$ is the force applied to the medium per unit length along the vertical axis z, per unit mean mass (i.e., the force per unit length of the momentum source, normalized by the mean density).

We stress that the reduction of the full system of equations of hydrodynamics and heat and admixture transport to the linear system of ordinary differential equations (78)–(80) relies solely on the symmetry of the problem, without resorting to the smallness of the perturbation amplitude (however, we must admit that the stability of the solution to be found may depend on its amplitude).

Far from the source (the z axis), obviously, all perturbations should decay. For $r \to 0$ a horizontal momentum flux exists, defined by the source in Eqn (78) (i.e., the applied force),

$$2\pi r\nu\,\frac{\mathrm{d}w}{\mathrm{d}r}=-P \quad \text{at} \quad r\to 0\,, \tag{81}$$

whence in can be concluded that w diverges logarithmically in approaching the axis. The horizontal fluxes of heat and admixture should apparently approach zero at $r = 0$.

Equation (78) connects the vertical velocity w with the dimensionless buoyancy $b \equiv \alpha\theta - \beta S$. Another equation connecting these variables can be obtained by taking a linear combination of Eqns (79) and (80) with the respective coefficients α/κ and $-\beta/\chi$:

$$\frac{\mathrm{d}^2 b}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}b}{\mathrm{d}r}-\frac{N^2}{g\kappa}\,w=0\,. \tag{82}$$

Here, $N^2 = N_T^2 + N_S^2/\tau$, $\tau = \chi/\kappa$, and $N_T^2 = \alpha g\gamma_T$ and $N_S^2 = -\beta g\gamma_S$ are squares of the 'thermal' and 'haline' buoyancy (Brunt–Väisälä) frequencies. Multiplying Eqn (78) by $N/g\sqrt{\kappa\nu}$ and Eqn (82) by the imaginary unit and combining them, we reduce the system to a single complex-valued equation for the variable $\eta = (N/g)\sqrt{\nu/\kappa}w + \mathrm{i}b$:

$$\frac{\mathrm{d}^2 \eta}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\eta}{\mathrm{d}r}-\mathrm{i}\,\frac{\eta}{L^2}=-\frac{P}{gL^2}\,\frac{\delta(r)}{2\pi r}=-\frac{\sigma}{L^2}\,\frac{\delta(r)}{2\pi r}\,, \tag{83}$$

where $L = (\nu\kappa/N^2)^{1/4}$ is the length scale. The solution of Eqn (83) can be expressed in terms of cylindrical functions. We write the result:

$$w=\frac{P}{2\pi\nu}\,\ker\frac{r}{L}\,,\qquad b\equiv\alpha\theta-\beta S=\frac{\sigma}{2\pi L^2}\,\mathrm{kei}\,\frac{r}{L}\,,$$

$$\theta=\frac{\gamma_T P}{2\pi N\sqrt{\kappa\nu}}\,\mathrm{kei}\,\frac{r}{L}\,,\qquad S=\frac{\gamma_S P}{2\pi N\tau\sqrt{\kappa\nu}}\,\mathrm{kei}\,\frac{r}{L}\,,$$

where ker and kei are the Kelvin functions [75].

Figure 15 presents normalized radial dependences of the vertical velocity for different variants of fluid stratification but with the same applied force. An ascending jet flow with the characteristic velocity $P/2\pi\nu$ is formed along the axis of the vertical momentum source (a compensating descending flow develops on the jet periphery). The dependence of the jet characteristic width L on the contributions to stratification from the tracers is nontrivial. Curve *1* corresponds to a one-component medium with a stable temperature stratification, $N_S^2 = 0$ and $N_T^2 = \Omega^2 > 0$, where Ω is a positive constant of the appropriate dimensionality. Curve *2* is related to the case of a neutral density stratification in which a stable contribution of salinity compensates the unstable one of the temperature, $N_S^2 = -N_T^2 = \Omega^2 > 0$. The radius r in the figure is normalized by L in this case [the coefficient of salinity diffusion is chosen to be 100 times smaller than that of thermal conductivity [1] ($\tau = 10^{-2}$)]. Curve *3* also corresponds to a fluid of uniform density, but the coefficients γ_T and γ_S are 10 times larger than in the preceding case.

If we pass from a stably stratified medium to a homogeneous one, it seems natural to expect that the latter can be involved in vertical motion more easily. But the comparison of curves *1* and *2* in Fig. 15 shows just the contrary. A medium with the neutral density stratification exhibits more resistance to the vertical force and is less involved in the vertical motion. This is a manifestation of the double diffusion effect we are exploring. Temperature perturbations caused by vertical fluid

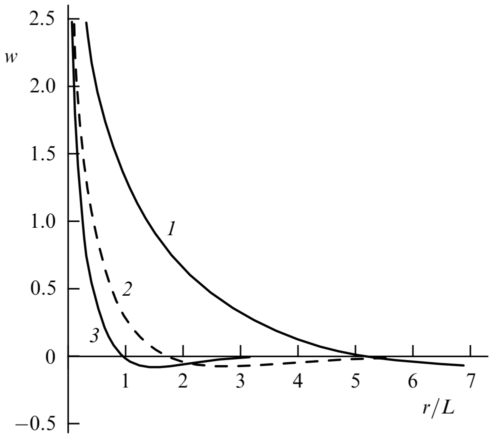

Figure 15. Radial dependences of vertical velocity (normalized by $P/2\pi\nu$) for different variants of medium stratification.

motions relax relatively rapidly, whereas longer-lived perturbations of salinity counteract vertical motions; this can be seen by comparing curves *2* and *3* with *1*. For an unstable salinity stratification, the opposite effect is possible when hydrodynamic drag is anomalously low (the given source of momentum can strongly influence vertical flows).

10. Convection: nonlinear interaction between fields of two tracers

Certain classes of interesting phenomena accompany convection in two-component media [76–80]. The velocity of developing convective motions depends on perturbations of buoyancy, i.e., a linear combination of perturbations of two components (temperature and salinity). The evolving flows, in turn, advect the substances, producing backaction on the buoyancy perturbation. This facilitates the nonlinear interaction between both fields. For example, convection of heat invokes the transport of an admixture, whose contribution to the buoyancy in turn modifies the velocity of flows, i.e., heat transport and the temperature distribution. For a set of problems with a sufficiently symmetric geometry, this interaction can be described in a closed analytic form [73, 76–79]. Such, for example, are problems pertaining to buoyant jets and isolated thermals in binary mixtures [76, 78].

One of the varieties of the aforementioned problems is the interaction between the two substances via their influence on turbulence [79]. As an example, we consider turbulent convection between two horizontal plates. The strength of turbulent motions (which can be described in the framework of the semi-empirical theory of turbulence, for example, in terms of turbulent exchange coefficients) depends on the vertical density gradient. This gradient, as well as the gradients of both components, in turn depends on turbulent motions. Thus, owing to the turbulent exchange, the gradient of temperature influences the transport of the admixture, and vice versa. Explicit analytic expressions for a related nonlinear problem in a particular, rather simple, closure scheme are obtained and analyzed in [79].

11. Conclusions

The rather simple analysis presented here indicates that certain general hydrodynamic features of two-component media can essentially differ from those in fluids whose density depends only on a single substance (temperature). Binary mixtures support previously unexplored mechanisms of convective instability, phenomena of hydrodynamic 'memory' and 'negative heat capacity,' the formation of discontinuities (jumps) of temperature, anomalous hydrodynamic drag, specific nonlinear interactions between their substances in convection, anomalously intense thermocapillary effects, and others.

The effects of this kind owe their existence, in particular, to the possibility of combining the following conditions in two-component media:

(i) an unstable temperature stratification but stable density stratification;

(ii) a strong (both stable and unstable) temperature stratification but weak density stratification;

(iii) intense vertical motions in the presence of a strong stable temperature stratification and the related strong vertical heat transfer.

More concisely, the hydrostatic stability depends on the vertical density gradient, whereas the vertical heat flux depends on the vertical temperature gradient. In a two-component medium, these gradients are not unambiguously connected, which makes a number of nontrivial effects possible.

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